A structure and representations of diffeomorphism groups of non-Archimedean manifolds. *

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Abstract

In this article diffeomorphism groups G of manifolds M on locally $\mathbf{F}$-convex spaces over non-Archimedean fields $\mathbf{F}$ are investigated. It is shown that their structure has many differences with the diffeomorphism groups of real and complex manifolds. It is proved that G is not a Banach-Lie group, but it has a neighbourhood W of the unit element e such that each element g in W belongs to at least one corresponding one-parameter subgroup.

It is proved that G is simple and perfect. Its compact subgroups G_c are studied such that a dimension over $\mathbf{F}$ of its tangent space $dim_{\mathbf{F}} T_e G_c$ in e may be infinite. This is used for decompositions of continuous representations into irreducible and investigations of induced representations.

1 Introduction.

This article is devoted to the investigation of a structure and representations of diffeomorphism groups of non-Archimedean manifolds. In previous

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works [18, 22] quasi-invariant measures on diffeomorphism groups relative to dense subgroups were constructed. Irreducible representations associated with the quasi-invaraint measures on groups and the corresponding configuration spaces were constructed in [18, 20]. Classical diffeomorphism groups (that is, for real or complex manifolds) play very important role in hydrodynamics, quantum mechanics and superstring theory [2, 9, 14]. On the other hand, non-Archimedean quantum mechanics develops rapidly [15, 16, 35]. It is helpful in special situations, when series or integrals divergent in quantum mechanics over the complex field $\mathbf{C}$ are convergent in the non-Archimedean case. In particular, non-Archimedean diffeomorphism groups can be used in non-Archimedean quantum mechanics and quantum gravity [18, 35].

There are many principal differences between classical and non-Archimedean functional analysis [4, 29, 31]. This is the source why non-Archimedean diffeomorphism groups differ in many respects from that of classical one.

In [23] it was shown that classical diffeomorphism groups are simple and perfect, but proofs there are based on local connectedness, homotopies, the existence and the uniquiness of solutions of differential equations in spaces of functions of the class of smoothness $C(t)$ for $t < \infty$. In the non-Archimedean case even for the class of smoothness $C(\infty)$ there is not any uniquiness, because of locally constant additional terms. In the classical case the small inductive dimension $ind(G) > 0$ (for real manifolds $ind(G) = \infty$), but in the non-Archimedean case $ind(G) = 0$. Therefore, the proof of simplicity and perfectness in this paper differ principally from the classical case. For compact complex manifolds the diffeomorphism groups are Lie groups [17], but in the non-Archimedean case, as it is proved below, it is untrue.

This article is devoted to more general diffeomorphism groups than in [18, 20]. Here are considered manifolds not only on Banach spaces over local fields, but also on locally $\mathbf{F}$-convex spaces, where $\mathbf{F}$ is an infinite field of characteristic $char(\mathbf{F}) = 0$ with non-trivial non-Archimedean valuation. Classes of smoothness $C(t)$ of manifolds M considered below are $1 \le t \le \infty$ and also analytic $t = an$ such that they are certainly not less than that of G. In particular this encompasses the class of manifolds treated by rigid analytic geometry (see about it in [4, 12]). This geometry is helpful in non-Archimedean superstring theory and theory of homologies and cohomologies, but it is related with very narrow class of analytic functions [7]. It is also extremely restrictive for non-Archimedean functional analysis and quantum theory. Therefore, differentiable manifolds of classes $C(t)$ for $1 \le t \le \infty$

also are considered below. Historically spaces of classes $C(t)$ with $t \in \mathbf{N}$ had appeared in [31, 32] several years later after the use of analytic spaces and manifolds in [7, 34]. Schikhof had used difference quotients of functions, Tate had used a topology stronger than the Zariski topology.

For locally compact groups there is a theory of induced representations from subgroups [3, 11], but its development for non-locally compact groups meets serious problems, because the case of non-locally compact groups is more complicated [19]. In this article with the help of structural theorems of diffeomorphism groups induced representations are investigated.

In §2 definitions, notations and preliminary results are given. In §3 the structure of diffeomorphism groups $Diff(t, M)$ is studied, where $Diff(t, M) := Hom(M) \cap C(t, M \to M)$, $C(t, M \to N)$ is a manifold of $C(t)$-mappings from a manifold M into a manifold N over the same field $\mathbf{F}$. Besides classes $C(t)$ also classes $C_0(t)$ are considered over local fields $\mathbf{K}$. If $dim_{\mathbf{F}} M \ge \aleph_0$, then $C(t, M \to M)$ is of non-separable type over $\mathbf{F}$, but $C_0(t, M \to M)$ is of separable type, when $\mathbf{F} = \mathbf{K}$ and $dim_{\mathbf{K}} M \le \aleph_0$. Such groups $G(t, M) := Hom(M) \cap C_0(t, M \to M)$ are helpful for the construction of quasi-invariant σ-finite measures. The diffeomorphism group is investigated below as the topological group and as the manifold. It is proved that $Diff(t, M)$ are simple and perfect. Then its structure as a manifold is studied. Apart from manifolds M on locally convex spaces X over $\mathbf{R}$ [27] in the case of X over $\mathbf{F}$ the existence of clopen (closed and open) subgroup W in $Diff(t, M)$ is proved below such that for each $g \in W$ there exists a one-parameter subgroup $< g^z : z \in \mathbf{F} >$ to which g belongs. Nevertheless, it is proved that $Diff(t, M)$ are not Banach-Lie groups. In §3 also families of compact subgroups $\{G^n_{u,\mathbf{K}}\}$ of the group $G(t, M)$ are constructed such that $\bigcup_{n,u,\mathbf{K}} G^n_{u,\mathbf{K}}$ is dense in $G(t, M)$. In the particular case of the local field $\mathbf{F} = \mathbf{K}$ such subgroups have the following property: the $\mathbf{K}$-linear span $sp_{\mathbf{K}}(T_e G^n_{u,\mathbf{K}})$ of $T_e G^n_{u,\mathbf{K}}$ is dense in $T_e G(t, M)$. This is the important difference from the case of M on X over $\mathbf{R}$ or $\mathbf{C}$, because the maximal compact subgroup in $G(t, M)$ in the classical case may be only finite-dimensional for finite-dimensional X over $\mathbf{R}$ or $\mathbf{C}$ [33]. This also is impossible in the classical case, when M is not a compact complex manifold. Embeddings of classical groups into the diffeomorphism groups also are discussed, because, for example, $Sp(2n, \mathbf{F})$ is very important for symplectic structures associated with Hamiltonians in quantum mechanics.

In §4 continuous unitary representations and also representations in non-

Archimedean Banach spaces are decomposed into irreducible. Then induced representations are considered. Moreover, two theorems (inductive-reductive and for internal tensor product representations) about decompositions of induced representations are proved. This opens new classes of unitary representations.

2 Topologies of non-Archimedean diffeomorphism groups.

To avoid misunderstandings we first present our definitions and notations in §§2.1-2.4.

2.1. Remarks. Let $\mathbf{K}$ be a local field, that is, a finite algebraic extension of the field $\mathbf{Q_p}$ of p-adic numbers and either $0 \le t < \infty$ or $t = an$. Then $C_*(t, M \to N)$, $Diff(t, M)$, $G(t, M)$ and $GC(t, M)$ be the same spaces as in [18, 20], where M and N are the corresponding Banach manifolds over $\mathbf{K}$, where either $* = \emptyset$ ($\emptyset$ is omitted as the index) or $* = 0$ or $* = c$. It is necessary to mention, that in in the case of M with an infinite atlas spaces in [18] are proper subspaces of the corresponding spaces in [20]. Then analogously we get these spaces for the class of locally analytic functions with $t = la$. Evidently, these spaces are isomorphic for different choices of atlases $At(M)$ and $At(N)$ for M and N of classes not less, than either $C(t)$ or $C_0(t)$ respectively, since the valuation group $\Gamma_{\mathbf{K}} := \{|x| : 0 \ne x \in \mathbf{K}\}$ is discrete in $(0, \infty)$ and due to [21] and Lemma 7.3.6 [10] each atlas of M or N has a disjoint covering $At'(M)$ or $At'(N)$, which is a refinement of the initial covering. Indeed, if $\phi : M \to M'$ and $\psi : N \to N'$ are $C_*(t)$-diffeomorphisms (that is, ϕ is bijective and surjective and $\phi \in C_*(t, M \to M')$, $\phi^{-1} \in C_*(t, M' \to M)$, analogously for ψ), then $g \mapsto \psi \circ g \circ \phi^{-1}$ is a diffeomorphism of $C_*(t, M \to N)$ with $C_*(t, M' \to N')$, where $g \in C_*(t, M \to N)$.

2.2. Notation. Let $\mathbf{F}$ be an infinite field of characteristic $char(\mathbf{F}) = 0$ with a non-trivial non-Archimedean valuation. For $b \in \mathbf{R}$, $0 < b \le 1$, we consider the following mapping:

$$(1)\ j_b(\zeta) := \zeta^b \in \mathbf{\Lambda_p} \text{ for } \zeta \ne 0,\ j_b(0) := 0,$$

such that $j_b(*) : \mathbf{F} \to \mathbf{\Lambda_p}$, where $\mathbf{\Lambda_p}$ is a spherically complete field with a valuation group $\{|x| : 0 \ne x \in \mathbf{\Lambda_p}\} = (0, \infty) \subset \mathbf{R}$ such that $\mathbf{C_p} \cup \mathbf{F} \subset \mathbf{\Lambda_p}$,

$\mathbf{C_p}$ denotes the field of complex numbers with the valuation extending that of $\mathbf{Q_p}$ [8, 29, 31, 36]. For a space X with a metric d in it let $B(X,y,r) := \{x \in X : d(x,y) \le r\}$ and $B(X,y,r^-) := \{x \in X : d(x,y) < r\}$ denote balls in X, where $0 < r$.

2.3. Definitions and Notes. Let us consider locally convex spaces X and Y over $\mathbf{F}$. Suppose $F : U \to Y$ is a mapping, where $U \subset X$ is an open bounded subset. The mapping F is called differentiable if for each $\zeta \in \mathbf{F}$, $x \in U$ and $h \in X$ with $x + \zeta h \in U$ there exists a differential such that

$$(1)\ DF(x,h) := dF(x+\zeta h)/d\zeta\ |_{\zeta=0} := \lim_{\zeta\to 0}\{F(x+\zeta h) - F(x)\}/\zeta$$

and $DF(x,h)$ is linear by h, that is, $DF(x,h) =: F'(x)h$, where $F'(x)$ is a bounded linear operator (a derivative). Let

$$(2)\ \Phi^b F(x;h;\zeta) := (F(x+\zeta h) - F(x))/j_b(\zeta) \in Y_{\mathbf{\Lambda_p}}$$

be partial difference quotients of order b for $0 < b \le 1$, $x + \zeta h \in U$, $\zeta h \ne 0$, $\Phi^0 F := F$, where $Y_{\mathbf{\Lambda_p}}$ is a locally convex space obtained from Y by extension of a scalar field from $\mathbf{F}$ to $\mathbf{\Lambda_p}$. By induction using Formulas $(1-2)$ we define partial difference quotients of order $n+b$ for each $0 < b \le 1$:

$$(3)\ \Phi^{n+b}F(x;h_1,...,h_{n+1};\zeta_1,...,\zeta_{n+1}) := \{\Phi^n F(x+\zeta_{n+1}h_{n+1};$$

$$h_1,..,h_n;\zeta_1,...,\zeta_n) - \Phi^n F(x;h_1,...,h_n;\zeta_1,...,\zeta_n)\}/j_b(\zeta_{n+1})$$

and derivatives $F^{(n)} = (F^{(n-1)})'$. Then $C(t,U\to Y)$ is a space of functions $F : U \to Y$ for which there exist bounded continuous extensions $\bar{\Phi}^v F$ for each x and $x + \zeta_i h_i \in U$ and each $0 \le v \le t$, such that each derivative $F^{(k)}(x) : X^k \to Y$ is a continuous k-linear operator for each $x \in U$ and $0 < k \le [t]$, where $0 \le t < \infty$, $h_i \in V$ and $\zeta_i \in S := B(\mathbf{K},0,1)$, $[t] = n \le t$ and $\{t\} = b$ are the integral and the fractional parts of $t = n+b$ respectively, U and V are open neighbourhoods of x and 0 in X, $U + V \subset U$. In the locally $\mathbf{F}$-convex space $C(t,U\to Y)$ its uniformity is given by the following family of pseudoultranorms:

$$(4)\ \|F\|_{C(t,U\to Y),u,w} := sup_{(x,x+\zeta_i h_i\in U;h_i\in V;u(h_i)\ne 0;\zeta_i\in S;i=1,...,s=[v]+sign\{v\};0\le v\le t)}$$

$$w\{(\bar{\Phi}^v F)(x; h_1, .., h_s; \zeta_1, ..., \zeta_s)\}/[\prod_{i=1}^s u(h_i)]^v$$

where $0 \le t \in \mathbf{R}$, $sign(y) = -1$ for $y < 0$, $sign(y) = 0$ for $y = 0$ and $sign(y) = 1$ for $y > 0$, $\{u\}$ and $\{w\}$ are families of pseudoultranorms in X and Y giving their ultrauniformities [25].

Then the locally $\mathbf{F}$-convex space

$$(5)\ C(\infty, U \to Y) := \bigcap_{n=1}^{\infty} C(n, U \to Y)$$

is supplied with the ultrauniformity given by the family of pseudoultranorms $\| * \|_{C(n,U\to Y),u,w}$.

2.4. Remarks. Spaces of analytic functions $C(an_R, B(X, x, R) \to Y)$ of radius of converegence not less than $0 < R$ are defined with the help of convergent series of polylinear polyhomogeneous functions [7] for normed spaces X and Y over $\mathbf{F}$. Spaces of locally analytic functions $C(la, M \to Y)$ are defined as inductive limits of spaces $C(la_r, M \to Y)$ of locally analytic functions f such that for each $x \in M$ there exists its neighbourhood U_x in M for which $f|_{U_x}$ has an analytic extension on $B(X, x, r)$, where $M \subset X$. Then using projective limits of normed spaces we can construct $C(la, M \to Y)$ for locally $\mathbf{F}$-convex spaces X and Y.

For $C(m)$-manifolds M and N on locally $\mathbf{F}$-convex spaces X and Y with atlases $At(M) = \{(U_i, \phi_i) : i \in \Lambda_M\}$ and $At(N) = \{(V_i, \psi_i) : i \in \Lambda_N\}$ a mapping $F : M \to N$ is called of class $C(t)$ if $F_{i,j}$ are of class $C(t)$ for each i and j, where $F_{i,j} = \psi_i \circ F \circ \phi_j^{-1}$, $\infty \ge m \ge t \ge 0$, $\phi_i : U_i \to \phi_i(U_i) \subset X$ and $\psi_l : V_l \to \psi_l(V_l) \subset Y$ are diffeomorphisms, U_i, V_l, $\phi_i(U_i)$ and $\psi_l(V_l)$ are open in M, N, X and Y respectively, $\phi_i \circ \phi_l^{-1} \in C(m, \phi_l(U_i \cap U_l) \to X)$ for each $U_i \cap U_l \ne \emptyset$, analogously for ψ_i.

Let $\pi_{z_1,..,z_n} : X \to sp_{\mathbf{F}}\{z_1, ..., z_n\}$ be a projection, where $z_1, ..., z_n$ are linearly independent vectors in X, then we set $C_0(t, M \to N)$ to be a completion of a subspace of cylindrical functions f of class $C(t)$, that is, for each such f there are $n \in \mathbf{N}$ and $z_1, ..., z_n$ linearly independent in X and $h \in C(t, (M \cap sp_{\mathbf{F}}\{z_1, ..., z_n\}) \to N)$ such that $f(x) = h(\pi_{z_1,...,z_n}(x))$. If $\theta : M \to N$ is a fixed mapping, then $C^\theta_*(M, Y)$ is a space of functions $f : M \to Y$ such that $(f - \theta) \in C_*(t, M \to Y)$, that induces a space $C^\theta_*(t, M \to N)$, where $* = \emptyset$ or $\theta = 0$.

Certainly we suppose throughout the paper, that M and N are of class $C(\tau)$ for spaces $C_*(t, M \to N)$ such that $\tau = \infty$ for $0 \le t \le \infty$, $\tau = an_r$ for $t = an_r$, $\tau = la$ for $t = la$.

Then $Diff(t, M) := Hom(M) \cap C^{id}(t, M \to M)$ and $G(t, M) := Hom(M) \cap C^{id}_0(t, M \to M)$ denote diffeomorphism groups for $t \ge 1$ or $t = an_R$ or $t = la$ and a homeomorphism group for $0 \le t < 1$ analogosuly to [20], where $Hom(M)$ is the standard homeomorphism group of $C(0)$ bijective surjective mappings of M onto itself, where the manifold M is on the locally $\mathbf{F}$-convex space X for $t \ne an_R$ and X is the normed space for $t = an_R$.

2.5. Let H be a locally $\mathbf{F}$-convex space, where $\mathbf{F}$ is a non-Archimedean field. Let M be a topological manifold modelled on H and $At(M) = \{(U_j, f_j) : j \in A\}$ be an atlas of M such that $card(A) \le w(H)$, where $f_j : U_j \to V_j$ are homeomorphisms, U_j are open in M, V_j are open in H, $\bigcup_{j\in A} U_j = M$, $f_i \circ f_j^{-1}$ are continuous on $f_j(U_i \cap U_j)$ for each $U_i \cap U_j \ne \emptyset$. Let $\tilde{\mathbf{F}}$, $\tilde{H}$ and $\tilde{M}$ denote completions of $\mathbf{F}$, H and M relative to their uniformities.

Theorem. *If either H is infinite-dimensional over $\mathbf{F}$, or $\tilde{\mathbf{F}}$ is not locally compact, then M is homeomorphic to the clopen subset of H.*

Proof. Since $\tilde{H}$ is the complete locally $\tilde{\mathbf{F}}$-convex space, then $\tilde{H} = pr - \lim\{\tilde{H}_q, \pi^q_v, \Upsilon\}$ is a projective limit of Banach spaces $\tilde{H}_q$ over $\tilde{\mathbf{F}}$, where $q \in \Upsilon$, Υ is an ordered set, $\pi^q_v : \tilde{H}_q \to \tilde{H}_v$ are linear continuous epimorphisms. Therefore, each clopen subset W in $\tilde{H}$ has a decomposition $W = \lim\{W_q, \pi^q_v, \Upsilon\}$, where $W_q = \pi^q_v(W)$ are clopen in $\tilde{H}_q$. The base of topology of $\tilde{M}$ consists of clopen subsets. If $W \subset V_j$, then $f_j^{-1}(W)$ has an analogous decomposition. From this and Proposition 2.5.6 [10] it follows, that $\tilde{M} = \lim\{\tilde{M}_q, \tilde{\pi}^q_v, \Upsilon\}$, where $\tilde{M}_q$ are manifolds on $\tilde{H}_q$ with continuous bonding mappings between charts of their atlases. If H is infinite-dimensional over $\mathbf{F}$, then each $\tilde{H}_q$ is infinite-dimensional over $\tilde{\mathbf{F}}$ [25]. From $card(A) \le w(H)$ it follows, that each $\tilde{M}_q$ has an atlas $At'(\tilde{M}_q) = \{U'_{j,q}; f_{j,q}; A'_q\}$ equivalent to $At(\tilde{M}_q)$ such that $card(A'_q) \le w(H_q) = w(\tilde{H}_q)$, since $w(\tilde{H}) = w(H)$, where $At(\tilde{M}_q)$ is induced by $At(\tilde{M})$ by the quotient mapping $\tilde{\pi}_q : \tilde{M} \to \tilde{M}_q$. In view of Theorem 2 [21] each $\tilde{M}_q$ is homeomorphic to a clopen subset $\tilde{S}_q$ of $\tilde{H}_q$, where $h_q : \tilde{M}_q \to \tilde{S}_q$ are homeomorphisms. To each clopen ball $\tilde{B}$ in $\tilde{H}_q$ there corresponds a clopen ball $B = \tilde{B} \cap H_q$ in H_q, hence $S_q = \tilde{S}_q \cap H_q$ is clopen in H_q and $h_q : M_q \to S_q$ is a homeomorphism. Therefore, M is homeomorphic to a closed subset V of H, where $h : M \to V$ is a homeomorphism, $V \subset H$, $h = \lim\{id, h_q, \Upsilon\}$, $id : \Upsilon \to \Upsilon$ is the identity mapping. Since

each h_q is surjective, then h is surjective by Lemma 2.5.9 [10]. If $x \in M$, then $\tilde{\pi}_q(x) = x_q \in M_q$, where $\pi : H \to H_q$ are linear quotient mappings and $\tilde{\pi}_q : M \to M_q$ are induced quotient mappings. Therefore, each $x \in M$ has a neighbourhood $\tilde{\pi}_q^{-1}(Y_q)$, where Y_q is an open neighbourhood of x_q in M_q. Therefore, $h(M) = V$ is open in H.

2.6. Theorems.

1. *The spaces $Diff(t, M)$, $G(t, M)$ and $GC(t, M)$ are the topological groups.*

2. *They have embeddings as clopen subsets into the spaces $C_*(t, M \to X)$, where either $* = \emptyset$ or $* = 0$ or $* = c$ respectively.*

3. *If $\mathbf{F}$ and X are complete, then $Diff(t, M)$, $G(t, M)$ and $GC(t, M)$ are complete.*

4. *$G(t, M)$ and $GC(t, M)$ are separable for separable M.*

5. *$Diff(t, M)$, $G(t, M)$ and $GC(t, M)$ are ultrametrizable for a manifold M with a finite atlas $At(M)$ on a normed space X and either $0 \le t < \infty$ or $t = an_r$.*

Proof. (A). Using the projective limits of normed spaces we can reduce the proof to the case of M on a normed space X, since for each continuous either linear mapping $A : X \to X$ or polylinear and polyhomogeneous mapping on X there are a pseudoultranorm u in X and a continuous mapping either linear ${}_uA(x + ker(u)) = A(x)$ or polylinear and polyhomogeneous ${}_uA(x_1 + ker(u), ..., x_n + ker(u)) = A(x_1, ..., x_n)$ from X_u into X_u, where $X_u := X/ker(u)$, $x, x_1, ..., x_n \in X$, $x + \ker(u) \in X_u$ (see Theorem (5.6.3) [25]). The second statement is the consequence of Theorem 2.5. If $f, g \in Diff(t, M)$ such that $0 < t$, then for each $0 < b \le \min(1, t)$ we have $(\Phi^b f \circ g)(x; \xi; h) = (\Phi^b f)(g(x); \zeta; z)$, where either $\zeta = \xi$ and $z = (\Phi^1 g)(x; \xi; h)$ for $b = 1$, or $\zeta \in \mathbf{F}$ and $z \in X$ such that $\zeta z = g(x + \xi h) - g(x)$ and $|\xi|^b/p \le |\zeta| \le |\xi|$, p is a prime number such that $\mathbf{Q_p} \subset \mathbf{F}$. In view of recurrence Relations 2.3.(3) we get that $Diff(t, M)$ is the topological group for each $0 \le t \le \infty$. In view of definitions $Diff(an_R, M)$ and $Diff(la, M)$ are also topological groups.

(B). Let at first $At(M)$ be finite. If $(f_n : n)$ is a Cauchy net in $C_*(t, M \to Y)$, then $(\Phi^v f_n : n)$ are uniformly convergent sequences for each $0 \le v \le t$ and $0 \le t \le \infty$, also for each v while $t = an_r$. Consequently, $\lim_{n \to \infty}(\Phi^v f_n) =: F^v \in C_*(\tau, M^{s+1} \to Y)$, where $\tau = 0$ for $0 \le t \le \infty$ or $\tau = an_r$ for $t = an_r$, $s := [v] + sign(\{v\})$.

The statement about ultrametrizability follows from §2.4 [20] and §2.2

[18]. If X is the Banach space, then from the completeness of $C_*(t, M \to Y)$, in which either $Diff(t, M)$ or $G(t, M)$ or $GC(t, M)$ respectively are closed, it follows that the latter spaces are also complete (see Theorems 8.3.6 and 8.3.20 [10]).

(C). In the case $G(t, M) \ni f, g$ for $0 \le t \le \infty$ due to §§2.1-2.4 [20] there is the equality

$$f_{i,j} \circ g_{j,l}(x) = \sum_{i \in I, n \in I, m \in \mathbf{N_o^n}} a(m, f^k_{i,j}) \bar{Q}_m((g_{j,l})_n(x)) q_i,$$

where $(g)_n = (g^{i(1)}, ..., g^{i(s)})$, $g_{j,l} = (g^k_{j,l}(x) : U_l \to \mathbf{K} | k \in I)$, M is modelled on $X = c_0(I, \mathbf{K})$, the set $\{i \in I : m(i) \ne 0\} = \{i(1), ..., i(s)\}$ is finite, $f_{i,j} = \phi_i \circ f \circ \phi_j^{-1}$ with the corresponding domains, $s \in \mathbf{N}$, $n = Ord(m)$,

$$\bar{Q}_m((g)_n) = \prod_{j=1}^{s} Q_{m(i(j))}(g^{i(j)}) \text{ and } Q_{m(i)}(g^i) := P_{m(i)}(g^i)/P_{m(i)}(u(m(i)),$$

where $P_{m(i)}$ are polynomials.

Coefficients $a(m, f^k_{i,j}) = \tilde{\Delta}^m(f^k_{i,j}(x))|_{x=0}$ are given by Corollary 2 from Proposition 7 [1]. The polynomials $[\bar{Q}_m(x) : |m| \le n, m(j) \ne 0 \text{ for } j \in (i(1), ..., i(s))]$ may be expressed throughout $[x^m : |m| \le n, m(j) \ne 0, \text{ for } j \in (i(1), ..., i(s))]$ and vice versa, where $x^m = \prod_{m(j) \ne 0} x(j)^{m(j)}$, $x(j) \in \mathbf{K}$, $x \in B(X, 0, 1)$. Therefore, $\tilde{\Delta}^m S_l(x)|_{x=0} = 0$ for each polynomial $S_l(x)$ with $l = (l(i) : i \in I, \mathbf{N_o} \ni l(i) \le m(i))$. Whence the coefficients $a(m, f^k_{i,j} \circ g_{j,l'})$ may be expressed throughout $a(l, f^k_{i,j}) a(q_{i(1)}, g^{i(1)}_{j,l'}) \ldots a(q_{i(s)}, g^{i(s)}_{j,l'}) R_{l,i,q}/P_l(\tilde{u}(l))$, where

$$(ii)\ k + |l| + Ord(l) + \sum_{j=1}^{s} (|q_{i(j)}| + Ord(q_{i(j)})) - s \ge k + |m| + Ord(m),$$

$q = (q_{(i(j))} \in \mathbf{N_o^{Ord(q_{i(j)})}} : j = 1, ..., s)$, $0 \le s \le |l|$, $R_{l,i,q}$ are polynomials by $u(i', j')$, that appear from the decomposition of $\bar{Q}_m((g_{j,l})_n)$ in the form of sums of products of $(g_{j,l})^k$ and $u(i', j')$ divided by $\bar{P}_m(\tilde{u}(m))$. In view of (i, ii) we get that $f \circ g \in G(t, M)$ and continuity of the composition, since in (ii) for $|m| + Ord(m) \to \infty$ or $|l| + Ord(l) \to \infty$ or there is $q_{i(j)}$ with $|q_{i(j)}| + Ord(q_{i(j)}) \ge [|m| + Ord(m) + 1]/s$. At the same time $s > 0$ for large $|m| + Ord(m)$. For $f = g^{-1}$ we get recurrence relations for $a(m, (f^{-1}_{i,j})^k)$ throughout $a(m, f^l_{i,j})$. From them follows that $\rho^t_0(f^{-1}, id)$ are

polynomials of the Bell type by $\rho_0^\kappa(f,id)$ in $1/p$ neighbourhood of id, where $\kappa = 0,1,...,[t], t < \infty$, ρ_0^t is an ultrametric in $G(t,M)$ (see also [20] and Chapter 5 [28]). This gives $f^{-1} \in G(t,M)$ and continuity of the inversion $f \to f^{-1}$. The case $t = \infty$ follows from Formula 2.3.(5).

(D). Now let $t = an_r$ and using the transformation $x \to x\xi$ with $|\xi| = 1/r$ we restrict the consideration to $r = 1$. If $g \in Diff(an_1, M)$ (or $G(an_1, M)$), then $\|g\| \le 1$. Indeed, there are the natural embeddings $\theta : B(\mathbf{K^n}, 0, 1) \to B(X, 0, 1)$, consequently, there are the restrictions $g|_{M_n} := g(\theta(x_n))$, where $\theta(x_n) = (x \in B(X,0,1) : \theta(x_n)(i) = x(j)$ for $i = i(j) \in (i(1),...,i(n)), \theta(x_n)(i) = 0$ in others cases $)$, $M_n = M \cap \theta(B(\mathbf{K^n}, 0, 1))$. In view of §54.4 in [31] with the help of [1] we get that if

$$f(x) = \sum_{m \in \mathbf{No^n}} a(m,f)\bar{Q}_m(x) \in C(0, B(\mathbf{K^n}, 0, 1) \to \mathbf{K}),$$

then f is analytic if and only if there exists

$$(iii) \quad \lim_{|m| \to \infty} a(m,f)/P_m(\tilde{u}(m)) = 0.$$

Moreover, in $C(an_1, B(\mathbf{K^n}, 0, 1) \to \mathbf{K})$ the following norms

$$(iv) \ \|f\| := \sup\{| \ a(m,f) \ | \ J(an,m) : m \in \mathbf{No^n}\} \quad \text{and}$$

$$(v) \ \|f\|" := \sup\{| \ b(m,f) \ |: m \in \mathbf{No^n}\}$$

are equivalent, where $J(an,m) :=| \ 1/P_m(\tilde{u}(m)) \ |$, $b(m,f)$ are expansion coefficients by x^m. Each function $g^k(\theta(x_n))$ is analytic and depends from a finite number of variables. If $\|g\| > 1$, so there is M_n with $\|g(\theta(x_n))\| > 1$.

The basis $\bar{Q}_m(x)$ is orthogonal in the non-Archimedean sense on $B(\mathbf{K^n}, 0, 1)$ with $\|\bar{Q}_m\|_{C(0,B\to\mathbf{K})} = 1$. Hence $|g^k(\theta(x_n))| > 1$ contradicts $g \in Hom(M)$ and $M \subset B(X,0,1)$. Therefore, $|a(m, g^k(\theta(x_n)))|J(an,m) \le 1$ for each k, n and such m, θ. Hence $\|g\|_{C_*(an_1,M\to M)} \le 1$, since θ has the natural extension $\theta : \mathbf{K^n} \hookrightarrow X$ such that θ is linear on $\mathbf{K^n}$ and it is the embedding. Therefore, the composition and the inversion operations are correctly defined and they are continuous in $Diff(an_1, M)$ and $G(an_1, M)$ due to Formulas (i, ii).

(E). Now let $At(M)$ be infinite. If $\mathbf{F}$, X and Y are complete, then $C_*(t, M \to Y)$ is complete (due to theorem about strict inductive limits in Chapter 12 [25]) for $t \ne la$. If $(f_\gamma : \gamma \in \alpha)$ is a Cauchy net in $C_*(la, M \to Y)$, consequently, there exist $\delta \in \alpha$, $E \in \Sigma$ and $r_0 > 0$ such that $supp(f_\gamma) \subset U^E$

and $f_\gamma \in C_*(an_{r_0}, M \to Y)$ for each $\gamma > \delta$, since $\Pi^r_R : C_*(an_R, U^E \to \mathbf{K}) \hookrightarrow C_*(an_r, U^E \to \mathbf{K})$ are compact operators for each $0 < r < R$, where α is a limit ordinal, Σ is a family of all finite subsets of Λ_M. From the completeness of $C_*(an_{r_0}, M \to Y)$ it follows that (f_γ) converges in $C_*(la, M \to Y)$, hence $C_*(la, M \to Y)$ is complete. From definitions it follows that $G(t, M)$, $Diff(t, M)$ and $GC(t, M)$ are closed in $C_*(t, M \to M)$ for $* = 0$, $* = \emptyset$ and $* = c$ respectively, whence they are also complete.

(F). For separable M and N the spaces $C_*(t, U^E \to N)$ are separable for each $E \in \Sigma$. The space $C_*(t, M \to Y)$ is isomorphic with the quotient space Z/P, where $Z = \bigoplus_{j \in \Lambda} C_*(t, U_j \to Y)$, P is closed and $\mathbf{K}$-linear in Z. From the separability of Z and $\Lambda \subset \mathbf{N}$ it follows that $C_*(t, M \to Y)$ is separable.

(G). From formulas (i,ii) it follows that $GC(t, M)$ is the topological group for M with the finite atlas. For f and $g \in C_*(t, M \to M) \cap Hom(M)$ for $0 \le t \le \infty$ or $t = an_r$ there are $E(f)$ and $E(g) \in \Sigma$ for which $supp(f) := cl\{x \in M : f(x) \ne x\} \subset U^{E(f)}$ and $supp(g) \subset U^{E(g)}$. Considering $f(supp(f))$ and $g(supp(g)) \subset M$ homeomorphic with $supp(f)$ and $supp(g)$ correspondingly we get $g^{-1} \circ f \in C_*(t, M \to M) \cap Hom(M)$. If $(f_\gamma : \gamma \in \alpha)$ and $(g_\gamma : \gamma \in \alpha)$ are two convergent nets in either $G(t, M)$ or $Diff(t, M)$ or $GC(t, M))$ to f and g respectively, so for each neighbourhood $W \ni id$ there exist $E \in \Sigma$ and $\beta \in \alpha$ such that $g_\gamma^{-1} \circ f_\gamma \in W \cap C_*(t, U^E \to M) \cap Hom(M)$ for $0 \le t \le \infty$ or $t = an_r$, where α is a limit ordinal. Therefore, for such t the mapping $(f, g) \to g^{-1} \circ f$ is continuous in $G(t, M)$ or $Diff(t, M)$ or $GC(t, M)$ respectively.

For $t = la$ let $r = \min(r(f), r(g))$, where f and $g \in C_*(la, M \to M) \cap Hom(M)$, that is, there exist $r(f)$ and $r(g) \in \Gamma_{\mathbf{F}}$ such that $f \in C_*(an_{r(f)}, M \to M) \cap Hom(M)$ and analogously for g. Then $r \in \Gamma_{\mathbf{F}}$ and $g^{-1} \circ f \in C_*(an_r, M \to M) \cap Hom(M)$. If $(f_\gamma : \gamma \in \alpha)$ converges to f and (g_γ) to g, then for each neighbourhood $W \ni id$ in $C_*(la, M \to M) \cap Hom(M)$ there exist $\beta \in \alpha$ and $E \in \Sigma$ such that $(supp(g_\gamma^{-1} \circ f_\gamma)) \cup (supp(g_\gamma)) \cup (supp(f_\gamma)) \subset U^E$ for each $\gamma > \beta$ and $r(g_\gamma) \ge r$, $r(f_\gamma) \ge r$. Therefore, $(g_\gamma \circ f_\gamma : \gamma \in \alpha)$ converges to $g^{-1} \circ f$ in $C_*(la, M \to M) \cap Hom(M)$, consequently, the last space is the topological group.

3 A structure of diffeomorphism groups.

3.1. Theorem. *Let the groups $G = Diff(t, M)$ and $G = G(t, M)$ be the*

same as in §2.4, where either $1 \le t \le \infty$ or $t = an_r$ or $t = la$.

(**1**). *If M is on a complete space X, then there exists a clopen subgroup W in G such that, each element $g \in W$ belongs to the corresponding one-parameter subgroup.*

(**2**). *$Diff(t,M)$, $G(t,M)$ and $GC(t,M)$ are not Banach-Lie groups.*

Proof. As in the proof of Theorem 2.6 we can use the projective limit $X = pr - \lim X_u$ of normed spaces X_u that reduce the proof to the case of the manifold M on the normed space X.

(**1**). Let at first $G = G(t,M)$ and M be with a finite atlas on X over a local field $\mathbf{K}$. We put $W := \{f \in G : \rho_0^\tau(f, id) \le p^{-2}\}$, then each $f \in W$ is an isometry of M, where $\tau = t$ for either $1 \le t \ne \infty$ or $t = an_r$, $\tau \in \mathbf{N}$ for $t = \infty$. If $f, g \in W$, then $\rho_0^\tau(f \circ g, id) = \rho_0^\tau(g, f^{-1}) \le \max(\rho_0^\tau(g, id), \rho_0^\tau(id, f^{-1})) = \max(\rho_0^\tau(g, id), \rho_0^\tau(f, id))$. Therefore, W is the subgroup in G.

Let at first M_n be finite-dimensional over $\mathbf{K}$. There exists a restriction $f|_{M_n}$ for each $f \in G$, where M_n is an analytic submanifold, $\theta : M_n \hookrightarrow M$ is an embedding, $dim_{\mathbf{K}} M_n = n \in \mathbf{N}$ is a dimension of M_n over $\mathbf{K}$. Since, locally polynomial functions $f(x) = id(x) + P(x)$ are dense in W, it is sufficient to prove that each such $f(x)$ belongs to a one-parameter subgroup. Here $deg P = m \in \mathbf{N}$ is a degree of a polynomial, $x \in M$ are a local coordinates. Denote $f_{i,j} = \phi_i \circ f \circ \phi_j^{-1}$ simply by f and U_j by M. Let

$$g(j;x) = \sum_{s=0}^{\infty} A(j;x)^s x/s!, \text{ where}$$

$$A(j;x) := \sum_{i=1}^{n} T(j,i;x)\partial_i,$$

$T(j,i;x)$ are polynomials on the j-th iteration, $A(j;x)^s x := A(j;x)(A(j;x)^{s-1}x)$ for $s > 1$, $A(j;x)^0 := x$, $\partial_i := \partial/\partial x^i$. Suppose $T(0,i;x) = P^i(x)$ for $i = 1, ..., n$, and $A(0;x)A(1;x) + A(1;x) = \bar{P}(x)$, where

$$P(x) = \sum_{i=1}^{n} P^i(x) e_i \text{ and } \bar{P}(x) = \sum_{i=1}^{n} P^i(x)\partial_i.$$

For the coefficients $T(1,i;x)$ there is the system of linear algebraic equations, that gives the unique solution $A(1;x)$ with

$$\|A(0;x)T(1,i;x)\|_\tau \le \|T(1,i;x)\|_\tau \times \|A(0;x)\|_\tau,$$

since $\|A(0;x)\| \le \|P(x)\|_\tau$, where

$$\|A(j;x)\|_\tau := \sup_{g\ne 0, g\in C_0(\tau,M\to X)} \|A(j;x)g\|_{\tau'}/\|g\|_\tau,$$

$\tau' = \tau - 1$ for $1 \le \tau < \infty$, $\tau' = \tau$ for $\tau = an_r$,

$$\|g\|_\tau := \|g\|_{C_0(\tau,M\to X)}.$$

Therefore,

$$\|P(x)\|_\tau = \max_{i=1,\dots,n} \|T(1,i;x)\|_\tau,$$

since $\|A(0;x)\|_\tau \le p^{-2}$. Moreover,

$$\max_{i=1,\dots,n} \|T(0,i;x) - T(1,i;x)\|_\tau \le \|P\|_\tau/p^2,$$

since $\|A(0;x)T(1,i;x)\|_\tau \le \|T(1,i;x)\|_\tau/p^2$ for each i.

Further by induction let for $j > 0$ are satisfied the following conditions:

$$(i)\ \bar{P}(x) = A(j;x) + \sum_{s=1}^{j} A(j-1;x)^s A(j;x)/s!,$$

$$(ii)\ \max_{i=1,\dots,n} \|T(j,i;x) - T(j-1,i;x)\|_\tau \le p^{-j}\|P\|_\tau \text{ and}$$

$$(iii)\ max_{i=1,\dots,n}\|T(j,i;x)\|_\tau = \|P\|_\tau.$$

For $j+1$ instead of j there exists the unique solution $A(j+1;x)$ of the equation (i), since $\bar{P}(x) = (I + S_{j+1})A(j+1,x)$ with $\|S_{j+1}\| \le 1/p$, I is the identity operator. To Equation (i) there corresponds the linear algebraic equation $(I_z + F)Z = Y$, Z and $Y \in \mathbf{K^z}$, $z \in \mathbf{N}$, I_z is the unit matrix and F is a matrix of size $z \times z$, $F = (F_{i,j})_{i,j=1,\dots,z}$, $F_{i,j} \in \mathbf{K}$, $\max_{i,j} |F_{i,j}| \le 1/p$, $|det(I_z + F)| = 1$. Then

$$\|A(j;x)^s T(j+1,i;x)/s!\| \le \|T(j+1,i;x)\| p^{-s(2-1/(p-1))} \text{ and}$$

$$t' := \max_{i=1,\dots,n} \|T(j+1,i;x)\| = \|P\|,$$

since $\|A(j;x)\| \le p^{-2}$. Consequently,

$$\|[A(j+1;x)^s - A(j;x)^s]/s!\| \le \|A(j+1;x) - A(j;x)\| p^z,$$

where $z=-(s-1)(2-1/(p-1))$, since $[A^i,B]=\sum_{l=0}^{i-1}A^l[A,B]A^{i-1-l}$, $\|[A,B]\|\le\max\{\|AB\|,\|BA\|\}$, $[A,B]:=AB-BA$, $A^i-B^i=A(A^{i-1}+A^{i-2}B+...+B^{i-1})-(A^{i-1}+A^{i-2}B+..+B^{i-1})B$, $\|AB\|\le\|A\|\times\|B\|$. Taking $A=A(j+1;x)$, $B=A(j;x)$ and using Formulas (ii,iii) we get $\|AB-BA\|\le\max\{\|AB-B^2\|,\|B^2-BA\|\}\le\|A-B\|/p^2$. From this it follows that

$$\max_{i=1,\dots,n}\|T(j+1,i;x)-T(j,i;x)\|\le$$

$$\max\{\|A(j;x)-A(j-1;x)\|t',\ \|A(j;x)^{j+1}A(j+1;x)/(j+1)!\|\}\le\|P\|p^z,$$

where $z=-(j+1)$, $t'=\|P\|$, since the second term in $\{,\}$ is less than the first and

$$\|T(j+1,i;x)-T(j,i;x)\|=\|P(i;x)-P(i;x)+\{\sum_{s=1}^{j-1}A(j-1;x)^s(T(j+1,i;x)-$$

$$T(j,i;x))/s!\}+\{\sum_{s=1}^{j-1}(A(j;x)^s-A(j-1;x)^s)T(j+1,i;x)/s!\}$$

$$+A(j;x)^{j+1}T(j+1,i,;x)/(j+1)!\|.$$

Therefore, there exists a sequence satisfying Formulas $(i-iii)$ for each j. Hence there exists

$$\lim_{j\to\infty}A(j;x)=A(x)$$

such that $A:C_0(\tau,M\to X)\to C_0(\tau',M\to X)$. This mapping may be considered as a vector field on M of class $C_0(\tau)$, $A(x)\in Vect_0(\tau,M)$, consequently, there exists

$$\lim_{j\to\infty}g(j;x)=g(x)\in C_0(t,M\to X).$$

In view of $\|A(j;x)^{s+j}/(s+j)!\|_\tau\le p^z$, where z= $-2(j+s)+(j+s)/(p-1)$ for each $s\in\mathbf{N}$ there is

$$exp\{qA(j;x)\}x=g^q(j;x)\text{ with }\lim_{j\to\infty}g^q(j;x)=g^q(x)\in W$$

(that is convergent relative to ρ_0^τ) for each $g(x)\in W$ and $q\in B(\mathbf{K},0,1)$ such that $g^1(x)=g(x)$. Moreover, to $\{g^q(x):q\in B(\mathbf{K},0,1)\}$ there corresponds

a one-parameter subgroup in W, where $q \in \mathbf{Z_p}$, since $\|qA(j;x)\|_\tau \le p^{-2}$ for each $q, y \in B(\mathbf{K}, 0, 1)$.

Indeed, $g^q = g^q_{i,j}$ are given as mappings from $\phi_j(U_j)$ into $\phi_i(U_i)$ for a given i, j, $\|g^q_{i,j} - id_{i,j}\|_\tau \le 1/p$, so $g^q_{i,j}$ generate $g^q \in W$, $g^q : M \to M$, since g^q is an isometry, consequently, $g^q :\in G(\tau, M)$. For $t = \infty$ we consider all $\tau \in \mathbf{N}$.

In general, for each $f \in G(t, M)$ there is a sequence $\{f_l(x) : l \in \mathbf{N}\} \subset G(t, M)$ such that in local coordinates $x = \{x(i) : i \in I\} \in B(c_0(I, K), 0, r)$ for each $i > l$ the following condition is satisfied $(f^i_l(x)) = x(i)$ and there exists $A_l(x) \in Vect_0(\tau, M)$ with the corresponding $g^q_l(x) \in W$ and $g^1_l(x) = f_l(x)$ for each $x \in M$, where $\lim_{j\to\infty} \|f_l - f\|_\tau = 0$. Then

$$\lim_{l\to\infty} g^q_l(x) = \lim_{l\to\infty} g^q(x) \in W$$

converges relative to ρ^τ_0 and $A(x) = \lim_{l\to\infty} A_l(x)$ with $\|A\|_\tau \le p^{-2}$, where

$$A = \sum_{m,i} a(m, A^i)\bar{Q}_m(x)\partial_i \in Vect_0(\tau, M),$$

$a(*, *) \in K$, that is, for each $c > 0$ the set $\{(i, Ord(m), |m|) : |a(m, A^i)|J(\tau, m) > c\}$ is finite.

The field $\mathbf{K}$ is equal to the disjoint union $\bigcup_{j\in\mathbf{N}} B(\mathbf{K}, k_j, 1)$, where $k_j \in \mathbf{K}$, $k_1 = 0$. Defining $g^{q+k_j} = g^q$ for $j > 1$ and $q \in B(\mathbf{K}, 0, 1)$, we get the extension of class $C_0(\tau)$ by q for g^q from $B(\mathbf{K}, 0, 1)$ onto $\mathbf{K}$ by q, for $1 \le t \le \infty$. For $t = an_r$ we use the additive group $B(\mathbf{K}, 0, 1)$. Then $\partial g^q(x)/\partial q = A(x)g^q(x)$ for each $q \in B(\mathbf{K}, 0, 1)$ and $x \in M$, $A = \sum_i A^i\partial_i$, $A^i \in C_0(\tau, M \to \mathbf{K})$.

In the cases of the non-local filed $\mathbf{F}$ or $G = Diff(t, M)$ consider the family $\Upsilon = \{\eta_{z_1,...,z_n,\mathbf{K}}\}$ of all embeddings $\eta_{z_1,...,z_n,\mathbf{K}} : sp_{\mathbf{K}}\{z_1, ..., z_n\} \hookrightarrow X$, where $\mathbf{K}$ are all possible local subfields of $\mathbf{F}$ and $z_1, ..., z_n$ are linearly independent vectors in X, $n \in \mathbf{N}$. If $f \in G$, then $f : M_{z_1,...,z_n,\mathbf{K}} \to f(M_{z_1,...,z_n,\mathbf{K}})$ is the diffeomorphism of class $C_0(t)$, where $M_{z_1,...,z_n,\mathbf{K}} := M \cap \eta_{z_1,...,z_n,\mathbf{K}}(sp_{\mathbf{K}}\{z_1, ..., z_n\})$. Let ρ^τ be a left-invariant ultrametric in G induced by the norm in $C(\tau, M \to X)$ for M with the finite atlas. There are embeddings of spaces $G(t, M_{z_1,...,z_n,\mathbf{K}})$ into G such that ρ^τ induces the equivalent ultrametric ρ^τ_0 in $G(t, M_{z_1,...,z_n,\mathbf{K}})$. Therefore, there exists a clopen subgroup W in G such that for each $f \in W$ and its restriction $f|_{M_{z_1,...,z_n,\mathbf{K}}}$ there exists a one-parameter family $\{g^q_{z_1,...,z_n,\mathbf{K}} : q \in \mathbf{K}\}$ which has an embedding into $W|_{M_{z_1,...,z_n,\mathbf{K}}}$. These families can be chosen consistent on $M_{z_1,...,z_n,\mathbf{K}} \cap M_{y_1,...,y_m,\mathbf{L}}$, since $\mathbf{K} \cap \mathbf{L}$ is a local field for two local fields $\mathbf{K}$ and

$\mathbf{L}$ such that $\mathbf{Q_p} \subset \mathbf{K} \cap \mathbf{L} \subset \mathbf{K} \cup \mathbf{L} \subset \mathbf{F}$, moreover, there exists a local field $\mathbf{J}$ such that $\mathbf{K} \cup \mathbf{L} \subset \mathbf{J}$. This means, that $g^q_{z_1,...,z_n,\mathbf{K}}(x) = g^q_{y_1,...,y_m,\mathbf{L}}(x)$ for each $x \in M_{z_1,...,z_n,\mathbf{K}} \cap M_{y_1,...,y_m,\mathbf{L}}$ and for each $q \in \mathbf{K} \cap \mathbf{L}$. Hence there exists $g^q(x)$ for each $x \in cl_M\{\bigcup_{z_1,...,z_n,\mathbf{K}} M_{z_1,...,z_n,\mathbf{K}}\}$ and each $q \in cl_{\mathbf{F}}\{\bigcup_{\mathbf{K}\subset\mathbf{F}} \mathbf{F}\}$, where $cl_M A$ denotes a closure of a subset A in M. In $\mathbf{C_p}$ the union of all local subfields is dense. If $\mathbf{F}$ is not contained in $\mathbf{C_p}$, then it can be constructed from $\mathbf{C_p}$ with the help of operations of spherical completion $\mathbf{C_p^U}$ or of quotients of definite algebras over $\mathbf{C_p}$ or $\mathbf{C_p^U}$ and so on by induction [8]. Therefore, these consisitent families generate a one-parameter subgroup $\{g^q : q \in B(\mathbf{F}, 0, 1)\}$ in W such that $g^1 = f$.

Let now M be with a countable infinite atlas and at first $1 \le t \le \infty$ then from the definition of topology in G the following set

$$W := \{f \in G : \ supp(f) \subset U^{E(f)}, \ E(f) \subset \mathbf{N}, card(E(f)) < \aleph_0, \rho^\tau_{0,U^{E(f)}}(f, id) \le p^{-2}\}$$

is the clopen subgroup, where $\rho^\tau_{0,U^E}(f, g)$ are ultrametrics in $G(t, U^E)$ inducing pseudoultrametrics in G, $\tau = t$ for $t \ne \infty$ and $1 \le \tau < \infty$ for $t = \infty$, $U^E := \bigcup_{i\in E} U_i$, (U_i, ϕ_i) are charts of M.

For $t = la$ let

$$W := \{f \in G : \ supp(f) \subset U^{E(f)}, E(f) \subset \mathbf{N}, card(E(f)) < \aleph_0,$$

$$\rho^{an_r}_{0,U^{E(f)}}(f, id) \le p^{-2}, f \in C_0(an_r, M \to M), r \in \Gamma_{\mathbf{F}}\},$$

where $\rho^{an_r}_{0,U^E}(f, g) := \sup_{i\in E} \|(g^{-1} \circ f - id)_{i,j}\|_{an,r,E}$.

(**2**). Let at first $t = an_1$. Let us prove that the function $exp : Vect(t, M) \to Diff(t, M)$ is not locally bijective. Let $M = B(\mathbf{F}, 0, 1)$ be a manifold over $\mathbf{F}$. We suppose, that there exists $q \in \mathbf{F}$ such that $q^l \ne 1$ for each $l = 1, ..., n-1$, $q^n = 1$, where n is not divisible by p and $1 < n \in \mathbf{N}$, $q^s \in \mathbf{F}$ and $| \ q^s \ |_p = 1$ for each $s \in \mathbf{Z_p} \cap \mathbf{F}$. Further $q^s M = M$ (acts as the multiplication $x \mapsto q^s x$ for each $x \in M$) and $q^s \in Diff(t, M)$, particularly, for $s = 1$, $(q^1)^n = id$. Let $H := \{g : g \in Diff(t, M), g^n = id\}$, consequently, $gq^1g^{-1} = q^1 = q$ for each $g \in H$. Whence q^1 belongs to each one-parameter subgroup gTg^{-1} in $Diff(t, M)$, where $T := \{q^s : s \in B(\mathbf{F}, 0, 1)\}$.

Now we consider the case, when the field $\mathbf{F}$ has not sufficient roots of unity. If G would be a Banach-Lie group, then there will exists a clopen subgroup W in G such that the Campbell-Hausdorff formula [5] will be valid. Let $g^q_m = exp(qx^m\partial)x = \sum_{l=0}^\infty (qx^m\partial)^n x/n!$, where $x \in M = B(\mathbf{F}, 0, 1)$, $0 \le m \in \mathbf{Z}$, $q \in$

$B(\mathbf{F}, 0, 1/p)$. Therefore, $g_0^q(x) = x + q$, $g_m^q(x) = \sum_{k=0}^{\infty} q^k x^{k(m-1)+1} \gamma(k,m)/k!$, where $\gamma(k,m) := m(2m-1)(3m-2)...((k-1)m-k+2)$ for $k \geq 1$, $\gamma(0,m) = 1$, $0! = 1$. Then

$$(ad\ u)^s(v) = \xi^s \zeta x^{n+s(m-1)}(n-m)(n-1)(n+m-2)...(n+(s-2)m-s+1)$$

for each elements $u = \xi x^m \partial$ and $v = \zeta x^n \partial \in \mathsf{g} := T_e G$, where $\xi, \zeta \in \mathbf{F}$, $u := \xi \partial$ for $m = 0$. Let $w = ln(e^u \circ e^v)$ be given by the Campbell-Hausdorff formula. Then calculating several lower terms of the series we get that $e^w(x)$ does not coincide with $g_n^\xi \circ g_n^\zeta(x)$, where $\xi, \zeta \in B(\mathbf{F}, 0, 1/p)$. This contradicts our supposition, consequently, G is not the Banach-Lie group. For $dim_{\mathbf{F}} X > 1$ it is sufficient to consider embeddings of $Diff(an_1, B(\mathbf{F}, 0, 1))^k$ into $Diff(an_r, M)$, where $1 < k \leq dim_{\mathbf{F}} X$.

In the case of $0 \leq t \leq \infty$ for each $f \in W$ there exists an infinite family g_l^q of one-parameter subgroups such that $g_l^1 = f$ and $\partial g_l^q/\partial q = \partial g_i^q/\partial q$ for each $i, l \in \mathbf{N}$, $q \in B(\mathbf{F}, 0, 1)$, since we can consider locally-constant additional terms for a given g^q.

Each subgroup $G(t, U^E)$ for $1 \leq t \leq \infty$ or $G(an_r, U_j)$ for $\phi_j(U_j) \subset B(X, \phi_j(x), r)$ are closed in G and are not the Banach-Lie groups , consequently, G is not the Banach-Lie group.

3.2. Theorem. *Let groups $G := Diff(t, M)$ and $G := G(t, M)$ be given by Definition 2.4. Then G is simple and perfect.*

Proof. It is sufficient to consider the case of a manifold M on a complete locally $\mathbf{F}$-convex space X, since the perfectness and simplicity of G and its completion $\tilde{G}$ are equivalent. Consider at first G with $t \geq 1$ or $t = an_r$. If $f, g \in W \subset G(t, M)$ (see Theorem 3.1), then there are vector fields A_f and A_g of class $C_0(t)$ on M and one-parameter subgroups f^q, $g^q \subset W$, $q \in B(\mathbf{F}, 0, 1)$ such that $\partial f^q/\partial q = A_f f^q$ and $\partial g^q/\partial q = A_g g^q$, where $A_f(x) = A_f^i(x)\partial_i$, the summation is accomplished by $i \in I$, I is an ordinal. Let A^i be of class $C_0(\tau)$ with $\tau = \infty$ for $1 \leq t \leq \infty$ or $\tau = t$ for $t = an_r$, then elements $exp(qA(x))x$ are dense in W for such t, where $A = A^i \partial_i$, $q \in B(\mathbf{K}, 0, 1)$. For $B = \bar{A}^i \partial_i$ with $\bar{A}^i = \delta_{i,j}$, where $j \in I$ is fixed, $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j \in I$, $W \ni exp(qB)x \neq id$. If $C \in Vect_0(\tau, M)$ with $\tau = \infty$ or $\tau = an_r$, also $\|C\|_{\tau'} \leq p^{-2}$ ($\tau' \in \mathbf{N}$ or $\tau' = an_r$ respectively), then there exists $A \in Vect_0(\tau, M)$ with $\|A\|_{\tau'} \leq p^{-2}$ such that their commutator $[A, B] = C$. Indeed, $[A, B] = A^i(\partial_i B^k \delta_{k,j})\partial_j - B^k \delta_{k,j}(\partial_j A^i)\partial_i = -(\partial_j A^i)\partial_i = C^i \partial_i$. In view of §79.1 and §80.3 [31] there is the antidifferentiation $P(x^j)$ by the variable x^j (in the

local coordinates of $At(M)$) such that $A^i(x) = -(P(x^j)C^i)(x)$. From this it follows that $[W, W] = W$, where $[W, W]$ is the minimal subgroup in W generated by the following subset $\{[g, f] := g \circ f \circ g^{-1} \circ f^{-1} | g, f \in W\}$. Therefore, W is perfect.

Suppose there is a normal subgroup V in W, $V \ne \{e\}$ and $V \ne W$, then $gVg^{-1} = V$ for each $g \in W$. Let $v \subset w$ be corresponding to V and W subsets in the algebra $Vect_0(\tau, M)$, hence $[v, w] \subset v$, where $[A, B]$ is the commutator in the algebra $Vect_0(\tau, M)$. Therefore, there are $A \in w \setminus v$ and $0 \ne B \in v$. Since $[v, w] \subset v$, then $[p^2\partial_i, B] \in v$ for each i, so it may be assumed that there is $i \in I$ with $a(0, B^i) \ne 0$.

For $Vect_0(an_r, M)$ we get the equations $\sum_{i,m+n=k+e_i} (b(n, C^i)b(m-e_i, B^j) - b(m, B^i)b(n - e_i, C^j)) = b(k, A^j)$, consequently, solving them recurrently we find $B \in v$ and $C \in w$ for which $[C, B] = A$. This is possible, since for each $c_l = p^{-l}$, $l \in \mathbf{N}$, the family $\{(i, |n|, Ord(n)) : |b(n, A^i)|r^{|n|} > c_l\}$ is finite for $A \in Vect_0(an_r, M)$, where $b(m, B^j) \in \mathbf{F}$ are expansion coefficients by $x^m := x_1^{m_1} ... x_n^{m_n}$, $x = (x_1, ..., x_n)$, $x_i \in \mathbf{F}$, $m = (m_1, ..., m_n)$, $0 \le m_i \in \mathbf{Z}$, $n \in \mathbf{N}$.

Locally analytic functions [31] are dense in $C_0(t, M \to X)$ for $1 \le t \le \infty$, hence, $[v, w] = w$ contradicting our assumption. Therefore, W is simple; that is, it does not contain any normal subgroup V with $V \ne \{e\}$ and $V \ne W$ at the same time.

The group G is the disjoint union of g_iW, $G = \bigcup_{j \in \mathbf{N}} g_jW$, such that $\rho_0^t(g_j, g_k) > p^{-2}$ for $j \ne k$, hence for chosen $\{g_j : j \in \mathbf{N}\}$ with $g_1 = e = id$ and each $f \in G \setminus W$ there is the unique j and $f_2 \in W$ with $f = g_jf_2$. From $\bar{Q}_m(x + c) = \bar{Q}_m(x)$ for $|m| > 0$ and each $x \in B(\mathbf{K}, 0, 1)$ it follows that $c = 0$, where $n = Ord(m)$, $\bar{Q}_m$ are basic Amice polynomials [1, 20]. Then considering all $g \in W$ having the form $(id + \xi\bar{Q}_m(x)e_i)$ in local coordinates with $\xi \in B(\mathbf{K}, 0, p^{-2})$ we get $[G \setminus W, W] \supset G \setminus W$ and $\{gfg^{-1} : g \in W\}$ $\ne \{f\}$ for each $f \in G \setminus W$, hence $[G, G] = G$.

Suppose there is a normal subgroup $V \subset G$, $V \ne e$, $V \ne G$. Then for each f and $g \in V$ with $fg^{-1} \in W$ we get that $fg^{-1} = e$, since $V \cap W$ is the normal subgroup in W, consequently, V is discrete and $\rho_0^t(f, g) > p^{-2}$ for each $f \ne g \in V$. Therefore, $hfh^{-1} = f$ for each $h \in W$, but this contradicts the statements given above. Therefore, G is simple and perfect.

For $0 \le t < 1$ the group $G(1, M)$ is dense in $G(t, M)$, for $t = la$ we use the inductive limit topology, consequently, $G(t, M)$ also is simple and perfect. The case of $Diff(t, M)$ follows from the case of $G(t, M)$ analogously to the

proof of Theorem 3.1.

3.3. Notes. For each chart (U_i, ϕ_i) there exists a tangent vector bundle $TU_i = U_i \times X$, consequently, TM has the following atlas $At(TM) = \{(U_i \times X, \phi_i \times I) : i \in \Lambda\}$, where $I : X \to X$ is the unit mapping, $\Lambda \subset \mathbf{N}$, TM is the tanget vector bundle over M.

Suppose V is a vector field on M. Then by analogy with the classical case we can define the following mapping $\bar{e}xp_x(zV) = x + zV(x)$, where for the corresponding pseudoultranorm u in X and sufficiently small $\epsilon > 0$ from $u(V(x))|z| \le \epsilon$ it follows $x + zV(x) \in U_i$ for each $x \in U_i$ such that $\phi_i(x)$ is also denoted by x, $z \in \mathbf{K}$. Moreover, there exists a refinement $At'(M) = \{(U'_i, \phi'_i) : i \in \Omega\}$ of $At(M)$ such that $\phi'_i(U_i)$ are $\mathbf{F}$-convex in X. The latter means that $\lambda x + (1-\lambda)y \in \phi'_i(U'_i)$ for each $x, y \in \phi'_i(U'_i)$ and each $\lambda \in B(\mathbf{F}, 0, 1)$. The atlas $At'(M)$ can be chosen such that $(\bar{e}xp_x|_{U'_i}) : x \times \{\phi_i(U'_i) - \phi_i(x)\} \to U'_i$ to be the analytic isomorphism for each $i \in \Omega$, where $x \in U'_i$.

Then $T_f C_*(t, M \to N) = \{g \in C(t, M \to TN)|\ \pi \circ g = f\}$, consequently, $C_*(t, M \to TN) = \bigcup_{f \in C_*(t,M\to N)} T_f C_*(t, M \to N)$, where $\pi : TN \to N$ is the natural projection. Therefore, the following mapping $\omega_{\bar{e}xp} : T_f C_*(t, M \to N) \to C_*(t, M \to N)$ such that $\omega_{\bar{e}xp}(g) = \bar{e}xp \circ g$ is defined. This gives charts on $C_*(t, M \to N)$ induced by charts in $C_*(t, M \to TN)$.

3.4. Theorems. *Let $G = Diff(t, M)$ and $G = G(t, M)$ be the same as in §2.4, where $1 \le t \le \infty$ or $t = la$, $\mathbf{F}$ and X are complete. Then*

(**1**) *if V is a $C(t)$-vector field on M, then its flow η_z is a one-parameter subgroup by $z \in B(\mathbf{F}, 0, 1)$ in G;*

(**2**) *the mapping $z \mapsto \eta_z$ is continuously differentiable by $z \in B(\mathbf{F}, 0, 1)$;*

(**3**) *the mapping $\tilde{E}xp : T_{id}G \mapsto G$, $V \mapsto \eta_z$, is continuous and defined in a neighbourhood of 0 in $T_{id}G$ for each $z \in B(\mathbf{F}, 0, 1)$, the mapping $(z, V) \mapsto \eta^V_z$ is of class $C(t)$;*

(**4**) *G is an analytic manifold and for it the mapping $\tilde{E} : TG \to G$ is defined such that $\tilde{E}_\eta(V) = \bar{e}xp_{\eta(x)} \circ V_\eta$ from some neighbourhood $\bar{V}_\eta$ of the zero section in $T_\eta G \subset TG$ onto some neighbourhood $W_\eta \ni id \in G$, such that $\bar{V}_\eta = \bar{V}_{id} \circ \eta$, $W_\eta = W_{id} \circ \eta$, $\eta \in G$ and $\tilde{E}$ belongs to the class $C(\infty)$ by V, $\tilde{E}$ is the uniform isomorphism of uniform spaces $\bar{V}$ and W.*

Proof. As in the proof of Theorem 3.1 we reduce the consideration to the case of M with a finite atlas on the Banach space X over $\mathbf{F}$ and $G = G(t, M)$ and then generalize results for infinite atlases on the locally $\mathbf{F}$-convex space X and $G = Diff(t, M)$ using inductive limits of spaces $C(t, U^E \to Y)$ and

the projective limit $X = pr - \lim X_u$.

For each submanifold M_n in M with the embedding $\theta : M_n \hookrightarrow M$ and $dim_{\mathbf{F}} M_n = n$ let us consider $V|_{M_n} : M_n \to TM$, $\pi \circ V(x) = x$. Therefore, in view of Theorem 6.1 [16] (and analogously we get existence of solutions in classes $C(t)$) there is the solution η_z for some $c > 0$ and all $z \in B(\mathbf{F}, 0, c)$, that is, $\partial \eta_z(x)/\partial z = V(x)\eta_z(x)$, $\eta_0(x) = x$ are dependent upon $x \in M$, $\eta_0 = id$, $\eta_z^V(x) = \eta_z(x)$ are dependent upon V. This local solution is unique in the analytic case, but it is not unique in $C(la)$ and $C(t)$ classes. Here a constant $\infty > c > 0$ depends only upon $0 < R < \infty$, M and t, where V is in the neighbourhood of the zero section $B(T_{id}C_*(\tau, M \to M), 0, R)$ and the ultranorm on the Banach space $T_{id}C_*(t, M \to M)$ is inherited from the Banach space $C_*(t, M \to TM)$.

The clopen ball $B(\mathbf{K}, 0, c)$ is the additive subgroup in $\mathbf{K}$, consequently, $z \mapsto \eta_z$ is the homomorphism such that $z_1 + z_2 \mapsto \eta_{z_1+z_2} = \eta_{z_1} \circ \eta_{z_2}$, $\eta_0 = id$. Moreover, $z \mapsto \eta_z$ and $V \mapsto \eta_z = \eta_z^V$ are C^∞-mappings by V and z in some neighbourhoods of 0. On the other hand, $B(\mathbf{F}, 0, 1)$ is a disjoint union of balls of radius $0 < c < 1$. Hence there exists a solution for each $z \in B(\mathbf{F}, 0, 1)$ (see aslo §45 in [31]).

Then there are R and c such that $\rho_*^t(\eta_z^V, id) \le 1/p$ for each $z \in B(\mathbf{F}, 0, c)$ and $V \in B(T_{id}C_*(t, M \to M), 0, R)$, hence for $Rc = R'c'$, $c' = 1$, we get the following mapping $V \mapsto \eta_z^V$ for each $V \in B(T_{id}C_*(t, M \to M), 0, R')$, where $z \in B(\mathbf{F}, 0, 1)$. Then $V \mapsto \eta_1$ generates the mapping $\tilde{E}xp(V) = \eta_1$. Hence, $\tilde{E}xp$ is defined in the neighbourhood of 0 in $T_{id}G$ and $\tilde{E}xp \in C^\theta(\infty, B(T_{id}G, 0, R') \to G)$, where the last space is given relative to the mapping $\theta = \tilde{\pi}_{id} : T_{id}G \to G$ being the natural projection.

Let $V(\eta) \in T_\eta G$ for each $\eta \in G$ and $V \in C_*(t, G \to TG)$, where $\tilde{\pi} \circ V(\eta) = \eta$, $\tilde{\pi} : TG \to G$, V is a vector field on G of class $C_*(t)$. If $\tilde{V} := \{g \in C_*(t, M \to M) : \rho_*^\kappa(g, id) \le 1/p\}$ and $g \in \tilde{V}$, where $\kappa = t$ for $t \ne \infty$ and $\mathbf{N} \ni \kappa \ge 1$ for $t = \infty$, then $g : M \to M$ is the isometry, consequently, $g \in Hom(M)$, that is, $\tilde{V} \subset G$ and G is a neighbourhood of id in $C_*(t, M \to M)$. Since $M = \bigcup_{i \in \Lambda} U_i$, $TM = \bigcup_i (U_i \times X)$, then $C_*(t, M \to M)$ and $C_*(t, M \to TM)$ have atlases with clopen charts. The C^∞-atlases $At(C_*(t, M \to M))$ and $At(C_*(t, M \to TM)$ induce clopen atlases in G and TG, since M and $\bar{e}xp$ are of class not less than C^∞ (see §2.4 and §3.3).

The right multiplication $R_f : G \to G$, $g \mapsto g \circ f = R_f(g)$ and $\alpha_h : C_*(t, M' \to N) \to C_*(t, M \to N)$, $\alpha_h(\zeta) = \zeta \circ h$ for $f \in G$ and $h \in C_*(t, M \to M')$ belong to the class $C(\infty)$ for $1 \le t \le \infty$, also to $C(an_r)$ for $t = an_r$ (see

Theorems 2.6). Let $g \in \tilde{V}$, then $g = id + Y$ with $\|\phi_i(Y|_{U_j})\|_{C_*(t,U_j \to X)} \le 1/p$ for each j (see §2.4), hence, $\tilde{g}_z = id + zY \in \tilde{V}$ for each $z \in B(\mathbf{F}, 0, 1)$ and $(\partial R_f \tilde{g}_z/\partial z)|_{z=0} = R_f Y$. From this it follows that each vector field V of class $C_*(t)$ on G is right-invariant, that is, $R_f V_\eta = V_{\eta \circ f}$ for each f and $\eta \in G$, since G acts on the right on G freely and transitively (that is, $g \circ f = f$ is equivalent to $f = id$, $Gf = fG = G$).

Therefore, the vector field V on G of class $C_*(t)$ has the form $V_{\eta(x)} = v \circ \eta(x) = v(\eta(x))$, where v is a vector field on M of the class $C_*(t)$, $\eta \in G$, $v(x) = V(id(x))$. Since $\bar{e}xp : TM \to M$ is locally analytic on corresponding charts, then $\tilde{E}(V) = \bar{e}xp \circ V$ has the necessary properties (see for comparison also the classical case in [2] and in §3, §9 [9]).

3.5. Notation. Let the group $G = G(t, M)$ be given by §2.4, where $\mathbf{Q_p} \subset \mathbf{F} \subset \mathbf{C_p}$, an atlas of M is countable. A complete locally $\mathbf{F}$-convex space X has a structure of a locally $\mathbf{K}$-convex space $X_\mathbf{K}$ over a local subfield $\mathbf{K}$ in $\mathbf{F}$, then $X_{u,\mathbf{K}} = X_\mathbf{K}/ker(u)$ is isomorphic with the Banach space $X_{u,\mathbf{K}} = c_0(J_u, \mathbf{K})$ over a local field $\mathbf{K}$, where J_u is an ordinal, u is a pseudoultranorm in X (see [25], Ch. 5 in [29], [20]). There exists $M_\mathbf{K}$ which is a manifold M on $X_\mathbf{K}$ instead of X. The projection $\pi_u : X_\mathbf{K} \to X_{u,\mathbf{K}}$ induces a projection $\pi_u : M_\mathbf{K} \to M_{u,\mathbf{K}}$, where $M_{u,\mathbf{K}}$ is a manifold on $X_{u,\mathbf{K}}$. Let each $X_u := X/ker(u)$ be of separable type over $\mathbf{F}$ for a family of pseudoultranorms $\{u\}$ defining topology of X. Let us consider the following space

$$C_{0,a}(t, M_{u,\mathbf{K}} \to X_{u,\mathbf{K}}) := \{f \in C_0(t, M_{u,\mathbf{K}} \to X_{u,\mathbf{K}}) : \ \|f\|_{t,a} :=$$
$$\sup_{k,i,j,m} [|a(m, \phi_k \circ f^i|_{U_j})| J_j(t, m) \max(p^i, p^{Ord(m)+|m|})] < \infty,$$
$$\lim_{j+i+|m|+Ord(m) \to \infty} |a(m, f^i_{U_j})| J_j(t, m) \max(p^i, p^{|m|+Ord(m)}) = 0\}$$

for $t \ne \infty$, $C_{0,a}(\infty, M_{u,\mathbf{K}} \to X_{u,\mathbf{K}}) := \bigcap_{l \in \mathbf{N}} C_{0,a}(l, M_{u,\mathbf{K}} \to X_{u,\mathbf{K}})$, where (U_j, ϕ_j) are charts of $At(M_{u,\mathbf{K}})$ with omitted index $(u, \mathbf{K})$, $J_j(t, m) := \|\bar{Q}_m|_S\|_{C_0(t, S \to X_{u,\mathbf{K}})}$, $S := (U_j)_{u,\mathbf{K}} \cap sp_\mathbf{K}\{e_1, ..., e_{Ord(m)}\}$, $\{e_j : j\}$ is the standard orthonormal base of $c_0(J_u, \mathbf{K})$.

For the manifolds M and N with a given mapping $\theta : M \to N$ (see §2.4) we define an ultrauniform space

$$C^\theta_{0,a}(t, M_{u,\mathbf{K}} \to N_{v,\mathbf{K}}) := \{f \in C^\theta_0(t, M_{u,\mathbf{K}} \to N_{v,\mathbf{K}}) | (f_{i,j} - \theta_{i,j}) \in C_{0,a}(t, \phi_j(U_j) \to Y_{v,\mathbf{K}})$$
$$\text{for each } i \text{ and } j, \text{ where } \rho^t_a(f, \theta) := \sum_{i,j} \|(f - \theta)_{i,j}\|_{C_{0,a}(t, \phi_j(U_j) \to Y_{v,\mathbf{K}})} < \infty\}$$

for each $0 \le t < \infty$. There exists a subgroup

$$G_a(t, M_{u,\mathbf{K}}) := G(t, M_{u,\mathbf{K}}) \cap C^{id}_{0,a}(t, M_{u,\mathbf{K}} \to M_{u,\mathbf{K}})$$

with an ultrametric $\rho^t_a(f, id)$ for $\theta = id$ and $0 \le t < \infty$.

3.6. Theorems. *Let* X, $\mathbf{F}$, $G := G(t, M)$ *and* $G_a(t, M_{u,\mathbf{K}})$ *be given by §2.4 and §3.5. Then*

(**1**). *For each* $0 \le t \le \infty$ *spaces* $G_a(t, M_{u,\mathbf{K}})$ *and* $C_{0,a}(t, M_{u,\mathbf{K}} \to X_{u,\mathbf{K}})$ *are separable and complete.*

(**2**). *Each* $G_a(t, M_{u,\mathbf{K}})$ *is* σ*-compact and* $G^r_a(t, M_{u,\mathbf{K}}) := B(G_a(t, M_{u,\mathbf{K}}), id, r) := \{g \in G_a(t, M_{u,\mathbf{K}}) | \ \rho^t_a(g, id) \le r\}$ *has an embedding as a compact separable subgroup in* $G(t, M)$ *in the topology inherited from it for* $0 \le t < \infty$ *and* $0 < r < \infty$.

(**3**) $T_eG(t, M) \subset Vect_0(t, M)$ *(see §3.2), moreover,* $sp_{\mathbf{K}} \bigcup_u T_eG_a(t, M_{u,\mathbf{K}})$ *and* $sp_{\mathbf{K}} \bigcup_u \ T_eB(G_a(t, M_{u,\mathbf{K}}), id, r)$ *are contained in* $T_eG(t, M_{\mathbf{K}})$ *and dense in it for* $1 \le t \le \infty$.

(**4**). *In* G *for* $0 \le t < \infty$ *there is a family* $\{G^n_{u,\mathbf{K}} : n \in \mathbf{N}, u, \mathbf{Q_p} \subset \mathbf{K} \subset \mathbf{F}\}$ *of compact subgroups such that* $G^n_{u,\mathbf{K}} \subset G^{n+1}_{u,\mathbf{K}}$ *for each* n *and each local subfield* $\mathbf{K}$ *in* $\mathbf{F}$*, moreover,* $\bigcup_{n\in\mathbf{N},u,\mathbf{K}} G^n_{u,\mathbf{K}} =: \bar{G}_a(t, M)$ *is dense in* G.

Proof. From Formulas 2.6(i, ii) it follows that $G_a(t, M_{u,\mathbf{K}})$ are the complete topological groups and $C_{0,a}(t, M_{u,\mathbf{K}} \to X_{u,\mathbf{K}})$ is the complete locally $\mathbf{K}$-convex space (and it is the Banach space for $0 \le t < \infty$). They are separable and Lindelöf, since $M_{u,\mathbf{K}}$ and $X_{u,\mathbf{K}}$ are separable.

The uniformity in $G_a(t, M_{u,\mathbf{K}})$ is given by the right-invariant ultarmetric $\rho^t_a(f, g) = \rho^t_a(g^{-1}f, id)$ for $t \ne \infty$ and by their family $\{\rho^l_a : l \in \mathbf{N}\}$ for $t = \infty$, where f and $g \in G_a(t, M_{u,\mathbf{K}})$. Therefore, $B(G_a(t, M_{u,\mathbf{K}}), id, r) =: G^r_{u,\mathbf{K}}$ is also the topological group, which is clopen in $G_a(t, M_{u,\mathbf{K}})$. The Banach space $C_0(t, M_{u,\mathbf{K}} \to X_{u,\mathbf{K}})$ is linearly topologically isomorphic with $c_0(\omega_{\mathbf{0}}, \mathbf{K})$ and subsets $S := \{x \in c_0(\omega_0, \mathbf{K}) : |x(i)|_{\mathbf{K}} \le p^{-k(i)}$ for each $i \in \mathbf{N}\}$ are sequentially compact in $c_0(\omega_0, \mathbf{K})$ for $\lim_{i\to\infty} k(i) = \infty$, consequently, S are compact [10]. In addition $sp_{\mathbf{K}}S$ is dense in $c_0(\omega_0, \mathbf{K})$.

Analogously to the proof of Theorems 3.4 we get neighbourhoods $\tilde{U} \ni 0$ in $T_{id}G_a(t, M_{u,\mathbf{K}})$ and $\tilde{W} \ni id$ in $G_a(t, M_{u,\mathbf{K}})$, such that $\tilde{E}|_{\tilde{U}} : \tilde{U} \to \tilde{W}$ is the uniform isomorphism. There exists an embedding of $G^r_{u,\mathbf{K}}$ into $G(t, M)$, since each function f on $M_{u,\mathbf{K}}$ has an extension $\tilde{f}$ on $M_{\mathbf{K}}$ such that $\tilde{f}|_{M_{\mathbf{K}} \ominus M_{u,\mathbf{K}}} = id$, where the decomposition $M_{\mathbf{K}} = (M_{\mathbf{K}} \ominus M_{u,\mathbf{K}}) \oplus M_{u,\mathbf{K}}$ is induced by the projection π_u, since $\mathbf{K}$ is spherically complete. Due to $\tilde{W} \supset G^r_{u,\mathbf{K}}$ for

$0 < r \le p^{-2}$ the subgroup $G^r_{u,\mathbf{K}}$ is compact in the weaker topology inherited from $G(t,M)$. For $p^{-2} < r < \infty$ considering in local cooordinates basic functions $\bar{Q}_m e_i$ and using the ultrametric ρ^t_a we get for $f \in G^r_{u,\mathbf{K}}$ that only a finite number of (m,k,i,j) are such that $|a(m, \phi_k \circ f^i|_{U_j})| > p^{-2}$. Therefore, $G^r_{u,\mathbf{K}}$ is compact. From $G_a(t, M_{u,\mathbf{K}}) = \bigcup_{i\in\mathbf{N}} g_i G^r_{u,\mathbf{K}}$ for some family $\{g_i : i \in \mathbf{N}\} \subset G_a(t, M_{u,\mathbf{K}})$ (or $G_a(t, M_{u,\mathbf{K}}) = \bigcup_{l\in\mathbf{N}} G^l_{u,\mathbf{K}}$) it follows, that $G_a(t, M_{u,\mathbf{K}})$ is σ-compact in $G(t,M)$.

In view of §3.2 and §3.4 we get $T_eG(t,M) \subset Vect_0(t,M)$ and $T_eG^r_{u,\mathbf{K}} \subset T_eG_a(t, M_{u,\mathbf{K}}) \subset T_eG(t, M_{\mathbf{K}})$, where $e = id$. In addition $G^r_{u,\mathbf{K}}$ contains all mappings f such that $\phi \circ f(x)|_{U_j} = (id(x) + c'\bar{Q}_m(x)e_i)$ with $n = Ord(m) \in \mathbf{N}$, $m \in \mathbf{N^n_o}$, $i \in \mathbf{N}$, $c' \in \mathbf{K}$ and $\rho^t_a(f, id) \le r$ (that is, for sufficiently small $|c'|_{\mathbf{K}}$ there is satisfied the following inequality $\|c'\bar{Q}_m\|_{C_{0,a}(t,U_j \to X_{u,\mathbf{K}})} \le p^{-2}$). Therefore, the closure in $Vect_0(t,M)$ of the $\mathbf{K}$-linear span of $\bigcup_u T_eG^r_{u,\mathbf{K}}$ coincides with $T_eG(t, M_{\mathbf{K}})$, which follows from Kaplansky Theorem A.4 [31]. Evidently, $T_eG^r_{u,\mathbf{K}}$ is infinite-dimensional over $\mathbf{K}$.

Let us take $G^n_{u,\mathbf{K}} := \{f \in G : supp(f) \subset U^{E_n}, f|_{U^{E_n}} \in C_{0,a}(t, U^{E_n} \to M_{u,\mathbf{K}}),\ \rho^t_a(f|_{U^{E_n}}, id) \le n\}$, where $U^E := \bigcup_{j\in E} U_j$, $E_n = (1,...,n)$, $n \in \mathbf{N}$, since $M_{u,\mathbf{K}}$ is separable. Each subgroup $G^n_{u,\mathbf{K}}$ is compact in G. Since $\bigcup_{n\in\mathbf{N}}\{f \in G : supp(f) \subset U^{E_n}\}$ is dense in G, then $\bar{G}_a(t,M) := \bigcup_{n,u,\mathbf{K}} G^n_{u,\mathbf{K}}$ is dense in G. If $f,\ g \in \bar{G}_a(t,M)$, then there exists n with $supp(f) \cup supp(g) \subset U^{E_n}$ and $g^{-1} \circ f \in \bar{G}_a(t,M)$, hence $\bar{G}_a(t,M)$ is the subgroup in G.

3.7. Remarks. If $M = B(X,0,1)$ for a normed space X, then $G_a(t, M_{u,\mathbf{K}})$ is a projective limit of discere groups $G_a(t, M_{u,\mathbf{K}})/B(G_a(t, M_{u,\mathbf{K}}), e, p^{-l})$ of polynomial bijective surjective mappings $\tilde{f} : k^n \to k^n$ of finite rings $k = B(\mathbf{K},0,1)/B(\mathbf{K},0,p^{-l})$, since a series $f(x) = \sum_{m,i} a(m,f^i)\bar{Q}_m(x)e_i$ in the Amice base $\bar{Q}_m e_i$ for each $f \in C^{id}_{0,a}(t, M_{u,\mathbf{K}} \to M_{u,\mathbf{K}})$ produces a finite sum $\tilde{f}(\tilde{x}) = \sum_{m,i} \tilde{a}(m,f^i)\tilde{Q}_m(\tilde{x})e_i$, where $\pi_l : \mathbf{K} \to \mathbf{K}/B(\mathbf{K},0,p^{-l})$ is the quotient mapping, $x \in B(\mathbf{K},0,p^{-l})$, $\tilde{x} = \pi_l(x)$, $a(m,*) \in \mathbf{K}$, $\tilde{a}(m,*) = \pi_l(a(m,*))$, $\tilde{Q}_m(\tilde{x}) = \pi_l(\bar{Q}_m(x))$, $l \in \mathbf{N}$ and $n \in \mathbf{N}$ depends on l, balls in $G_a(t, M_{u,\mathbf{K}})$ are given relative to the ultrametric ρ^t_0 in $G(t,M)$. If $X_{u,\mathbf{K}}$ is finite-dimensional over $\mathbf{K}$, then n are bounded by $dim_{\mathbf{K}} X_{u,\mathbf{K}}$.

Besides profinite subgroups given in §3.6 there are classical subgroups over the non-Archimedean field $\mathbf{F}$ contained in the diffeomorphism groups. In particular subgroups preserving vector fields are important for quantum mechanics. Let M be an analytic manifold, which is is a clopen subset in $B(X,0,r)$, where $r > 0$, and $X = \mathbf{F^n}$. For a covector field $\tilde{A} := \{A_\alpha(x) :$

$\alpha = 1, ..., n\}$ on M a differential 1-form

(i) $A = A_\alpha(x)dx^\alpha$ is called a potential structure, where summation is by $\alpha = 1, ..., n$. It is called analytic if $\tilde{A} \in C(an_r, M \to \mathbf{F^n})$. It is called non-degenerate, if

(ii) $det(F_{\alpha,\beta}) \ne 0$ for each x, where

(iii) $F = dA = F_{\alpha,\beta}dx^\alpha \Lambda dx^\beta/2$. We consider $g \in Diff(an_r, M)$ and $y^\alpha = g^\alpha(x)$, $x = (x^1, ..., x^n) \in M$, $x^i \in \mathbf{F}$. If

(iv) $A_\alpha = A_\mu \partial g^\mu/\partial x^\alpha$ or

(v) $F_{\alpha,\beta} = F_{\mu,\nu}(\partial g^\mu/\partial x^\alpha)(\partial g^\nu/\partial x^\beta)$. The groups of such g are denoted by G_A or G_F respectively and are called a potential group or a symplectic group respectively. Corresponding to them Lie algebras of vector fields are denoted by $\mathsf{L_A}$ and $\mathsf{L_F}$. There are accomplished analogs of Proposals 1 and 2 [26], since $G_A \subset G_F$ and $\mathsf{L_A} \subset \mathsf{L_F}$. Let $n = 2m$, $m \in \mathbf{N}$ and

(vi) $A = c_{\alpha,\nu}x^\nu dx^\alpha$, where $c_{\alpha,\nu} = -c_{\nu,\alpha} = const$ and $det(c_{\alpha,\nu}) \ne 0$; or

(vii) $A = A_\alpha dx^\alpha$, $A_\alpha = \sum_{k=1}^\infty c^k_{\alpha,\nu_1,...,\nu_k} x^{\nu_1}...x^{\nu_k}$, where $c^k_{\alpha,\nu_1,...,\nu_k} \in \mathbf{F}$, $c^1_{\alpha,\nu} = -c^1_{\nu,\alpha}$ for each $\alpha, \nu = 1, ..., N$, $det(c^1_{\alpha,\nu}) \ne 0$. Then

$(viii)$ $dim_{\mathbf{F}}\mathsf{L_A} = n(n+1)/2$, $G_A = Sp(2m, \mathbf{F}) := \{g \in GL(2m, \mathsf{F}) |\ g^t \epsilon g = \epsilon\}$ is the symplectic group, where g^t denotes the transposed matrix;

(ix) $dim_{\mathbf{F}}\mathsf{L_A} \le n(n+1)/2$.

To verify this let us consider at first $c_{\alpha,\mu} = \epsilon_{\alpha,\mu}$, where $\epsilon_{\alpha,\alpha+1} = 1$, $\epsilon_{\alpha+1,\alpha} = -1$ for $\alpha = 1, ..., n-1$, $\epsilon_{\alpha,\beta} = 0$ for others (α, β). Therefore, there are true analogs of Formulas (10-13) [26] with $a^\mu_{\nu_1,...,\nu_k} \in \mathbf{F}$. The matrices $B^{(k),\sigma}_{\alpha,\nu,\mu}$ in Lemma 1 §2 have integer elements, consequently, there are true analogs of Formulas (15,16) for the field $\mathbf{F}$, since an analytic vector field ξ is in $\mathsf{L_A}$ if and only if $\xi^\mu \partial_\mu A_\alpha +\ A_\mu \partial_\alpha \xi^\mu =: L_\xi A_\alpha = 0$. In general the form A can be reduced to $A = -\lambda \epsilon_{\alpha,\nu} x^\nu dx^\alpha/2$ by some operator $j \in GL(n, \mathbf{F})$, where $\lambda \in \mathbf{F}$ and $j(B(\mathbf{F^n}, 0, 1)) = B(\mathbf{F^n}, 0, 1)$. Theorem 2 [26] can also be modified, but should be rather lengthy.

In $G(t, M)$ for $0 \le t \le \infty$ there are also subgroups isomorphic with the additive group $B(X_u, 0, r)$, elements f of which act as translations of a subset V of M diffeomorphic with $B(X_u, 0, r)$ and $f|_{M \setminus V} = id$. Using disjoint coverings of $M_{u,\mathbf{K}}$ by balls we get, that $Diff(t, M)$ contains subgroups isomorphic with symmetric groups S_n, where either $n \in \mathbf{N}$ or $n = \infty$ for non compact M. Also $Diff(t, M)$ contains a subgroup diffeomorphic with $W := \{f : f|_V$ has an extension $\tilde{f} \in GL(X_u)$ $\|\tilde{f} - I\|_{X_u} < 1$, $f|_{M \setminus V} = id\}$, where $GL(X_u)$ is the general linear group.

4 Decompositions of representations and induced representations.

4.1. Let $G = G(t, M)$ be defined as in §2.4 and §3.5 with $0 \le t < \infty$ and $T : G \to U(H)$ be a strongly continuous unitary representation, where $\mathbf{Q_p} \subset \mathbf{F} \subset \mathbf{C_p}$, $U(H)$ is a unitary group of a Hilbert space H over $\mathbf{C}$. The unitary group is in the standard topology inherited from the space $L(H)$ of continuous linear operators $A : H \to H$ in the operator norm topology.

Theorem. *The representation T up to the unitary equivalence $T \mapsto A^{-1}TA$ is decomposable into the direct integral $T_g = \int_J T_g(y)dv(y)$ of irreducible representations $G \ni g \mapsto T_g(y) \in U(H_y)$, where H_y are Hilbert subspaces of H, $y \in J$, v is a σ-additive measure on a compact Hausdorff space J, A is a fixed unitary operator.*

Proof. In view of Theorems 3.6 there exists a family of compact subgroups $G^n_{u,\mathbf{K}}$ in $(G, V(G))$ for which $G^n_{u,\mathbf{K}} \subset G^{n+1}_{u,\mathbf{K}}$ for each n and $N := \bar{G}_a(t, M)$ is dense in $(G, V(G))$, where $V(G)$ denotes the topology of G. Then

$$T_g|_{G^n_{u,\mathbf{K}}} = \int_{J(n,u,\mathbf{K})} T_g(n, u, \mathbf{K}; y)v_{n,u,\mathbf{K}}(dy)$$

for each $n \in \mathbf{N}$, a pseudoultranorm u in X and a local subfield $\mathbf{K} \subset \mathbf{F}$, where $v_{n,u,\mathbf{K}}$ are measures on compact spaces $J(n, u, \mathbf{K})$, $T_g(n, u, \mathbf{K}; y)$ are finite-dimensional irreducible representations, $y \in J(n, u, \mathbf{K})$, $g \in G^n_{u,\mathbf{K}}$.

There is the consistent family $T_g(n, u, \mathbf{K}; y)$ such that $v_{n,u,\mathbf{K}}$-almost everywhere in $J(n + 1, u', \mathbf{K}')$ the restriction $T_g(n + 1, u', \mathbf{K}'; y)|_{G^n_{u,\mathbf{K}}}$ is a finite direct sum of $T_g(n, u, \mathbf{K}; y)$ with the corresponding y, where $u(a) \le u'(a)$ for each $a \in X$, $\mathbf{K} \subset \mathbf{K}'$. Therefore, there are continuous mappings $z(-n, u, \mathbf{K}; -n', u', \mathbf{K}')$ from $J(n, u, \mathbf{K})$ into $J(n', u', \mathbf{K}')$ for each $n < n'$, $u \le u'$ and $\mathbf{K} \subset \mathbf{K}'$ such that $v_{n',u',\mathbf{K}'}(Y) = v_{n,u,\mathbf{K}}(z^{-1}(-n, u, \mathbf{K}; -n', u', \mathbf{K}')(Y))$ for each Y in the Borel σ-filed $Bf(J(n', u', \mathbf{K}'))$, where $v_{n,u,\mathbf{K}}$ are non-negative measures. For each $\xi, \eta \in H$ with $\|\xi\| = \|\eta\| = 1$ we have $|(\xi, T_g y)| \le 1$ and

$$|\int_{J(n,u,\mathbf{K})} (\xi, T_g(n, u, \mathbf{K}; y)\eta)v_{n,u,\mathbf{K}}(dy)| \le 1.$$

Consequently, $T_g|_N = \int_J T_g(y)v(dy)$, where the projective limit of compact spaces $J = pr - \lim\{J(n, u, \mathbf{K}); z(-n, u, \mathbf{K}; -n', u'\mathbf{K}'); \{(n, u, \mathbf{K})\}\}$ is compact (see also §2.5 [10]) and the projective limit of measures $v = pr -$

$lim\{v_{n,u,\mathbf{K}}\}$ is the measure on $(J, Bf(J))$, and $T_g(y) : N \to U(H_y)$ are irreducible for v-almost every $y \in J$. Therefore,

$$T_g = \int_J T_g(y) v(dy) \text{ and } H = \int_J H_y v(dy),$$

where $T_g(y) : G \to U(H_y)$ is an irreducible unitary representation for v-almost each $y \in J$, H_y are complex Hilbert subspaces of H (see [24] and §22.8 [13]).

4.2. Let $G := G(t, M)$ be given by §§2.4 and 3.5, where $\mathbf{Q_p} \subset \mathbf{F} \subset \mathbf{C_p}$. Suppose $W : G \to IS(H)$ is the regular representation (for H over a local field $\mathbf{L} \supset Q_s$, $s \ne p$) given by the formula $U_g f(x) := f(g^{-1}x)$, where H is a Banach space of uniformly continuous bounded functions $f : G \to \mathbf{L}$ with a norm $\|f\| := \sup_{x \in G} |f(x)|_{\mathbf{L}}$, $IS(H)$ is a group of isometries of H with a metric induced by an operator norm of continuous $\mathbf{L}$-linear operators V, $V : H \to H$.

Theorem. *There exists $A \in IS(H)$ such that AWA^{-1} is decomposable into a direct sum of irreducible representations W_j. Moreover, each irreducible representation $T : G \to IS(E)$ for a Banach space E over $\mathbf{L}$ is equivalent to some W_j.*

Proof. It may be directly verified that W is strongly continuous. This means that for each $c > 0$ and $f \in H$ there is a neinghbourhood $V \ni id$ such that $\|W_g f - f\| \le c$ for each $g \in V$. Let the compact subgroups $G^n_{u,\mathbf{K}}$ be the same as in the proof of Theorem 4.1. They s-free, that is, for each clopen subgroup E' its index $[G^n_{u,\mathbf{K}} : E']$ is not divisible by s ([29, 30]). It follows from the consideration of local decompositions of elements in $G^n_{u,\mathbf{K}}$ by $\bar{Q}_m e_i$ and from the fact that $B(\mathbf{K}, 0, r)$ are the s-free additive groups for each $0 < r < \infty$, $n \in \mathbf{N}$. In addition E' contains an open compact subgroup which is normal in $G^n_{u,\mathbf{K}}$ due to Pontryagin lemma (see §8.1 [29]). Therefore, on $G^n_{u,\mathbf{K}}$ the Haar measure exists with values in $\mathbf{Q_s}$ due to Monna-Springer theorem 8.4[29]. In view of Theorem 2.6 and Corollary 2.7 [30] each strongly continuous representation $\tilde{T} : G^n_{u,\mathbf{K}} \to IS(H)$ is decomposable into the direct sum of irreducible representations. On the other hand, $\bar{G}_a(t, M)$ is dense in G. The last statement of this theorem follows from the fact that for compact groups each $T : G^n_{u,\mathbf{K}} \to IS(H_T)$ is equivalent to some irreducible component of the regular representation, where H_T is a Banach space over $\mathbf{L}$.

4.3. Remark. Let μ be a Borel regular Radon non-negative quasi-invariant measure on a diffeomorphism group G relative to a dense subgroup

G' with a continuous quasi-invariance factor $\rho_\mu(x,y)$ on $G' \times G$ and $0 < \mu(G) < \infty$ [18]. Suppose that $V : S \to U(H_V)$ is a strongly continuous unitary representation of a closed subgroup S in G'. There exists a Hilbert space $L^2(G,\mu,H_V)$ of equivalence classes of measurable functions $f : G \to H_V$ with a finite norm

$$(1)\ \|f\| := (\int_G \|f(g)\|^2_{H_V} \mu(dg))^{1/2} < \infty.$$

Then there exists a subspace Ψ_0 of functions $f \in L^2(G,\mu,H_V)$ such that $f(hy) = V_{h^{-1}} f(y)$ for each $y \in G$ and $h \in S$, the closure of Ψ_0 in $L^2(G,\mu,H_V)$ is denoted by $\Psi^{V,\mu}$. For each $f \in \Psi^{V,\mu}$ there is defined a function

$$(2)\ (T^{V,\mu}_x f)(y) := \rho^{1/2}_\mu(x,y) f(x^{-1}y),$$

where $\rho_\mu(x,y) := \mu_x(dy)/\mu(dy)$ is a quasi-invariance factor for each $x \in G'$ and $y \in G$, $\mu_x(A) := \mu(x^{-1}A)$ for each Borel subset A in G. Since $(T^{V,\mu}_x f)(hy) = V_{h^{-1}}((T_x f)(y))$, then $\Psi^{V,\mu}$ is a $T^{V,\mu}$-stable subspace. Therefore, $T^{V,\mu} : G' \to U(\Psi^{V,\mu})$ is a strongly continuous unitary representation, which is called induced and denoted by $Ind_{S\uparrow G'}(V)$.

If $S = \lim\{S_\alpha, \pi^\alpha_\beta, \Omega\}$ is a profinite subgroup of G, for example, $G^n_{u,\mathbf{K}}$ (see §§3.6, 3.7) and V is irreducible, then H_V is finite-dimensional and $V^{-1}(I) =: W$ is a clopen normal subgroup in S, where $\pi^\alpha_\beta : S_\alpha \to S_\beta$ are homomorphisms of finite groups S_α and S_β for each $\alpha \le \beta$ in an ordered set Ω. Therefore, there exists $\alpha \in \Omega$ such that $\pi^{-1}_\alpha(e_\alpha) \subset W$, where e_α is the unit element in S_α and $\pi_\alpha : S \to S_\alpha$ is a quotient homomorphism. In view of Theorems 7.5-7.8 [13] there exists a representation $V^\alpha : S_\alpha \to U(H_V)$ such that $V^\alpha \circ \pi_\alpha = V$.

4.4. Let G be a diffeomorphism group with a non-negative quasi-invariant measure μ relative to a dense subgroup G'. We can choose G' such that each $G^n_{u,\mathbf{K}}$ is a compact subgroup of G'. Suppose that there are two closed subgroups K and N in G such that $K' := K \cap G'$ and $N' = N \cap G'$ are dense subgroups in K and N respectively. We say that K and N act regularly in G, if there exists a sequence $\{Z_i : i = 0, 1, ...\}$ of Borel subsets Z_i satisfying two conditions:

(i) $\mu(Z_0) = 0$, $Z_i(k,n) = Z_i$ for each pair $(k,n) \in K \times N$ and each i;

(ii) if an orbit O relative to the action of $K \times N$ is not a subset of Z_0, then $\mathsf{O} = \bigcap_{Z_i \supset \mathsf{O}} Z_i$, where $g(k,n) := k^{-1}gn$. Let $T^{V,\mu}$ be a representation of G' induced by a unitary representation V of K' and a quasi-invariant measure

μ as in §4.3. We denote by $T^{V,\mu}_{N'}$ a restriction of $T^{V,\mu}$ on N' and by D a space $K \setminus G/N$ of double coset classes KgN.

Theorem. *There are a unitary operator A on $\Psi^{V,\mu}$ and a measure ν on a space D such that*

$$(1)\ A^{-1}T^{V,\mu}_{n}A = \int_{\mathsf{D}} T_n(\xi)d\nu(\xi)$$

for each $n \in N'$. (2). Each representation $N' \ni n \mapsto T_n(\xi)$ in the direct integral decomposition (1) is defined relative to the equivalence of a double coset class ξ. For a subgroup $N' \cap g^{-1}K'g$ its representations $\gamma \mapsto V_{g\gamma g^{-1}}$ are equivalent for each $g \in G'$ and representations $T_{N'}(\xi)$ can be taken up to their equivalence as induced by $\gamma \mapsto V_{g\gamma g^{-1}}$.

Proof. A quotient mapping $\pi_{\mathsf{X}} : G \to G/K =: \mathsf{X}$ induces a measure $\hat{\mu}$ on X such that $\hat{\mu}(E) = \mu(\pi_{\mathsf{X}}^{-1}(E)) =: (\pi^*_{\mathsf{X}}\mu)(E)$ for each Borel subset E in X. In view of Radon-Nikodym theorem II.7.8 [11] for each $\xi \in \mathsf{D}$ there exists a measure μ_ξ on X such that

$$(3)\ d\hat{\mu}(x) = d\nu(\xi)d\mu_\xi(x),$$

where $x \in \mathsf{X}$, $\nu(E) := (s^*\mu)(E)$ for each Borel subset E in D, $s : G \to \mathsf{D}$ is a quotient mapping.

For each $m \in \mathbf{N}$, a pseudoultranorm u in X and a local subfield $\mathbf{K}$ in $\mathbf{F}$ a subgroup $G^m_{u,\mathbf{K}}$ is compact in G, hence there exists a topological retraction $r_{m,u,\mathbf{K}} : G \to G^m_{u,\mathbf{K}}$ (that is, $r_{m,u,\mathbf{K}} \circ r_{m,u,\mathbf{K}} = r_{m,u,\mathbf{K}}$ and $r_{m,u,\mathbf{K}}$ is continuous and $r_{m,u,\mathbf{K}}|_{G^m_{u,\mathbf{K}}} = id$). This retraction induces a measure $\mu_{m,u,\mathbf{K}} = r^*_{m,u,\mathbf{K}}\mu$ on $G^m_{u,\mathbf{K}}$. It is equivalent to a Haar measure on $G^m_{u,\mathbf{K}}$, since it is quasi-invariant relative to $G^m_{u,\mathbf{K}}$ (see §VII.1.9 in [6]). In view of §26 [25] and Formula (3) the Hilbert space $H^V := L^2(\mathsf{X}, \hat{\mu}, H_V)$ has a decomposition into a direct integral

$$(4)\ H^V = \int_{\mathsf{D}} H(\xi)d\nu(\xi),$$

where H_V denotes a complex Hilbert space of the representation $V : K' \to U(H_V)$. Therefore,

$$\|f\|^2_{H^V} = \int_{\mathsf{D}} \|f\|^2_{H(\xi)}d\nu(\xi).$$

In view of §32.2 from Chapter VI [24] each irreducible representation of a compact group Y can be realized as a representaion in some minimal left

ideal of a ring $L^2(Y,\lambda,\mathbf{C})$, where λ is a Haar measure on Y. From Formulas (4) and 4.3.(1, 2) we get the first statement of this theorem for a subspace $\Psi^{V,\mu}$ of H^V.

If $f \in L^2(\mathsf{X},\hat{\mu},H_V)$, then $\pi^*_{\mathsf{X}}f := f\circ\pi_{\mathsf{X}} \in L^2(G,\mu,H_V)$. This induces an embedding π^*_{X} of H^V into $\Psi^{V,\mu}$. Let F be a filterbase of neighbourhoods A of K in G such that $A=\pi^{-1}_{\mathsf{X}}(S)$, where S is open in X, hence $0<\mu(A)\le\mu(G)$ due to quasi-invaraince of μ on G relative to G'. Let $\psi\in\xi\in\mathsf{D}$, then $\psi=Kg_\xi$, where $g_\xi\in G$, hence $\psi=\psi(N\cap g^{-1}_\xi Kg_\xi)$. In view of Formula (3) for each $x\in N'$ and $\eta=Kx$ we get $\rho^{1/2}_{\mu_\xi}(\eta,\xi)=\lim_{\mathsf{F}}[\int_A\rho^{1/2}(x,zg_\xi)d\mu(z)/\mu(A)]$ (see also §1.6 [10]). Therefore, $(a,T_x(\xi)b)_{H^V}=\lim_{\mathsf{F}}[\int_A(\pi^*a,\rho^{1/2}_\mu(x,zg_\xi)(\pi^*b)^{zg_\xi}_x)_{\Psi^{V,\mu}}d\mu(z)/\mu(A)]$ for each $x\in N'$ and $a,b\in H^V$, where $f^h_z(\zeta):=f(z^{-1}h\zeta)$ for a function f on G and $h,z,\zeta\in G$. In view of the cocycle condition $\rho_\mu(yx,z)=\rho_\mu(x,y^{-1}z)\rho_\mu(y,z)$ for each $x,y\in G'$ and $z\in G$ we get $T_{yx}(\xi)=T_y(\xi)T_x(\xi)$ for each $x,y\in N'$ and $T_x(\xi)$ are unitary representations of N'. Then $(a,T_{yx}(\xi)b)_{H^V}=\lim_{\mathsf{F}}[\int_A(\pi^*a,V_{g_\xi yg^{-1}_\xi}[\rho^{1/2}_\mu(x,zg_\xi)(\pi^*b)^{zg_\xi}_x])_{\Psi^{V,\mu}}d\mu(z)/\mu(A)]$ for each $y\in N'\cap g^{-1}_\xi K'g_\xi$. Hence the representation $T_x(\xi)$ in the Hilbert space $H(\xi)$ is induced by a representation $(N'\cap g^{-1}_\xi K'g_\xi)\ni y\mapsto V_{g_\xi yg^{-1}_\xi}$.

4.5. Let V and W be two unitary representations of K' and N' (see §4.4). In addition let K and N be regularly related in G and $V\hat{\otimes}W$ denotes an external tensor product of representations for a direct product group $K\times N$. In view of §4.3 a representation $T^{V,\mu}\hat{\otimes}T^{W,\mu}$ of an external product group $\mathsf{G}:=G\times G$ is equivalent with an induced representation $T^{V\hat{\otimes}W,\mu\otimes\mu}$, where $\mu\otimes\mu$ is a product measure on G. A restriction of $T^{V\hat{\otimes}W,\mu\otimes\mu}$ on $\tilde{G}:=\{(g,g): g\in G\}$ is equivalent with an internal tensor product $T^{V,\mu}\otimes T^{W,\mu}$.

Theorem. *There exists a unitary operator A on $\Psi^{V\hat{\otimes}W,\mu\otimes\mu}$ such that*

$$(1)\ A^{-1}T^{V,\mu}\otimes T^{W,\mu}A=\int_{\mathsf{D}}T(\xi)d\nu(\xi),$$

where ν is an admissible measure on a space $\mathsf{D}:=N\setminus G/K$ of double cosets.

(2). *Each representation $G'\ni g\mapsto T_g(\xi)$ in Formula (1) is defined up to the equivalence of ξ in D. Moreover, $T(\xi)$ is unitarily equivalent with $T^{\tilde{V}\otimes\tilde{W},\mu\otimes\mu}$, where $\tilde{V}$ and $\tilde{W}$ are restrictions of the corresponding representations $y\mapsto V_{gyg^{-1}}$ and $y\mapsto W_{\gamma y\gamma^{-1}}$ on $g^{-1}K'g\cap\gamma^{-1}N'\gamma$, $g,\gamma\in G'$, $g\gamma^{-1}\in\xi$.*

Proof. In view of §18.2 [3] $P\setminus\mathsf{G}/\tilde{G}$ and $K\setminus G/N$ are isomorphic Borel spaces, where $P=K\times N$. In view of Theorem 4.4 there exists a unitary

operator A on a subspace $\Psi^{V\hat{\otimes}W,\mu\otimes\mu}$ of the Hilbert space $L^2(\mathsf{G},\mu\otimes\mu,H_V\otimes H_W)$ such that

$$A^{-1}T^{V\hat{\otimes}W,\mu\otimes\mu}|_{\tilde{G}}A=\int_{\mathsf{D}}T_{\tilde{G}}(\xi)d\nu(\xi),$$

where each $T_{\tilde{G}}(\xi)$ is induced by a representation $(y,y)\mapsto(V\hat{\otimes}W)_{(g,\gamma)(y,y)(g,\gamma)^{-1}}$ of a subgroup $\tilde{G}'\cap(g,\gamma)^{-1}(K\times N)(g,\gamma)$, the latter group is isomorphic with $S:=g^{-1}K'g\cap\gamma^{-1}N'\gamma$, that gives a representation $\tilde{V}\hat{\otimes}\tilde{W}$ of a subgroup $S\times S$ in G. Therefore, we get a representation $T^{\hat{V}\hat{\otimes}\hat{W},\mu\otimes\mu}$ equivalent with $Ind_{(S\times S)\uparrow\mathsf{G}'}(\tilde{V}\hat{\otimes}\tilde{W})|_{\tilde{G}'}$.

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